OPTIMIZATION TECHNIQUES IN OPERATIONS RESEARCH

B. D. SIVAZLIAN

L. E. STANFEL

OPTIMIZATION TECHNIQUES IN OPERATIONS RESEARCH

B. D. SIVAZLIAN
University of Florida

L. E. STANFEL
University of Texas at Arlington

PRENTICE-HALL, INC., *Englewood Cliffs, New Jersey*

Library of Congress Cataloging in Publication Data

Sivazlian, B.D.
Optimization techniques in operations research.

(Prentice-Hall international series in industrial and systems engineering)
Companion volume to Analysis of systems in operations research.
Includes bibliographies.
1. Operations research. 2. Mathematical optimization. 3. Programming (Mathematics)
I. Stanfel, L. E., joint author.
II. Title.
T57.6.S59 001.4'24 73-18320
ISBN 0-13-638163-4

TO JANE

No dedication can match her own

10 9 8 7 6 5 4 3

Printed in the United States of America

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*

OPTIMIZATION TECHNIQUES IN OPERATIONS RESEARCH

PRENTICE-HALL INTERNATIONAL SERIES IN INDUSTRIAL AND SYSTEMS ENGINEERING

W. J. Fabrycky and J. H. Mize, Editors

BLANCHARD *Logistics: Engineering and Management*
FABRYCKY, GHARE, AND TORGERSEN *Industrial Operations Research*
FRANCIS AND WHITE *Facility Layout and Location: An Analytical Approach*
GOTTFRIED AND WEISMAN *Introduction to Optimization Theory*
KIRKPATRICK *Introductory Statistics and Probability for Engineering, Science, and Technology*
MIZE, WHITE, AND BROOKS *Operations Planning and Control*
OSTWALD *Cost Estimating for Engineering and Management*
SIVAZLIAN AND STANFEL *Analysis of Systems in Operations Research*
SIVAZLIAN AND STANFEL *Optimization Techniques in Operations Research*
THUESEN, FABRYCKY, AND THUESEN *Engineering Economy, 4/E*
WHITEHOUSE *Systems Analysis and Design Using Network Techniques*

PREFACE

The aim of this book is to provide an introduction to the principal methods of optimization in present day operations research. Together with its companion volume, *Analysis of Systems in Operations Research,* it provides comprehensive coverage for the undergraduate engineering or mathematics student of junior or senior level whose mathematical background is assumed to include elementary calculus and a modicum of differential equations. In addition, the book will serve as an introductory text for qualified students in such fields as business and economics, and for freshly entering graduate students in certain programs who have not had exposure to the techniques of operations research. The practicing operations researcher will also find the book a helpful reference.

It is the case that the preparation mentioned is most often realized by students during their second undergraduate year, and portions of the book will comprise a cohesive, understandable introduction to certain areas in operations research for the student of late sophomore standing. The authors feel that an early introduction of this sort is most desirable; for example, the pre-engineering student rarely receives a hint of the nature of this area of study, whereas other fields of engineering are represented by required courses during the first two years.

The book is not an entirely theoretical treatment of the various topics; first, this is not possible in terms of the background presumed and the audience being addressed; secondly, feasibility not withstanding, that sort of exposition would not serve as the undergraduate-level introductory treatment we hope to provide—namely, to impart a problem-solving facility to the students.

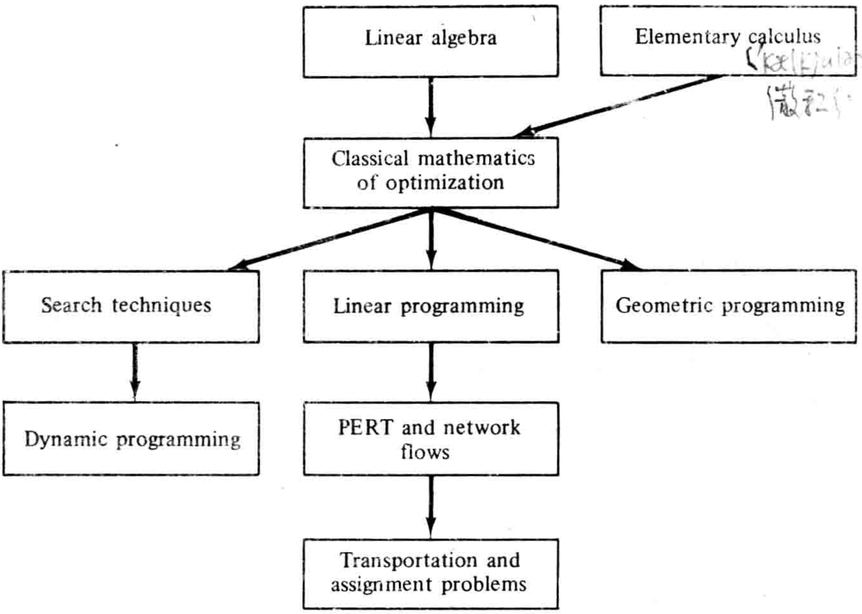
On the other hand, the treatment of a given topic does not consist of the presentation of a few algorithms and a collection of solved problems. This would be, the authors feel, meaningless from a pedagogical point of view, with respect to both the material under consideration and the preparation and stimulation of the students for possible further pursuit of the topics. The understanding of the book's scope and content is a first step for the ambitious student who will be delving into the more complex and specialized literature.

We have attempted to strike a useful balance between these two extremes. Numerous pertinent results have been derived in terms meaningful to the group being addressed, the more rigorous portion of this book, as well as its companion book, corresponding to those topics whose underlying foundations rest more heavily upon the students' presumed minimum background. Thus, for example, the chapter dealing with classical optimization will be found to be more rigorously developed than, say, the chapter on linear programming.

When the presentation becomes, of necessity, less rigorous—and in fact, whenever possible—we have attempted to provide insight by relying upon illustrations of results and a sufficient development to make a final result intuitively appealing and meaningful. We have frequently utilized geometry in two and three dimensions for illustrative purposes, and numerous examples have been included. Various computational aspects of the application of the techniques have been discussed, since the utility of an application often hinges upon these.

It is believed that both the set of *optimization techniques* and the *fields of application* constitute operations research. Thus, dynamic programming is an optimization technique, whereas inventory systems is a field in operations research in which such optimization techniques are useful. The two books are separated within this conceptual framework. The present text treats some of the primary optimization techniques, including classical mathematics of optimization, linear programming, PERT and network flows, transportation and assignment problems, search techniques, dynamic programming, and geometric programming. Its initial chapter provides the elementary concepts of linear algebra and matrix manipulations which are utilized in subsequent chapters, while the essentials of probability theory are presented at the beginning of *Analysis of Systems in Operations Research*. Students with no exposure to these are in no way disadvantaged.

Optimization Techniques in Operations Research



The book contains chapters not ordinarily found in texts of this kind. A chapter on geometric programming, a recent and—for certain problems—a most effective optimization technique is included, as well as one devoted to search techniques, a topic of much importance in any practical considerations of optimization problems.

In choosing notations we have attempted to select those that are fairly conventional within subject areas and which are, simultaneously, not unwieldy or cumbersome.

The problems at the end of each chapter will be found to include not only a number of the problem-solving variety, but also several requiring the proofs of statements relative to the development, extensions thereof, etc. The problems will assist the student in mastering certain fundamental techniques and the reader is urged to solve them, as they are believed to form an integral part of the text.

B. D. Sivazlian
L. E. Stanfel

CONTENTS

OPTIMIZATION TECHNIQUES IN OPERATIONS RESEARCH

LINEAR ALGEBRA

1.1. Introduction

It is the purpose of this chapter to provide the reader with a few fundamental concepts of linear algebra and matrices. The brief coverage should enable such a reader to undertake—without disadvantage—those portions of the book which require some familiarity with these concepts.

1.2. Vector Spaces

i. *Definitions*

Definition. A set of objects, called *vectors*, V, is said to be a vector space over the real numbers if the following conditions hold:

1. Sums of vectors from V form other vectors that belong to V.
2. Scalar multiples of vectors by real numbers form vectors that belong to V.

1

The Euclidean vector spaces E^n over the set of real numbers consist of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. Here the vector addition is defined as

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

Multiplication by a scalar c is defined as

$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$$

Vectors will be denoted by block symbols; thus

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

and

$$\mathbf{y} = (y_1, y_2, \ldots, y_n)$$

Example 1.1

$$(1, 2, 3) + (4, 5, 6) = (5, 7, 9).$$

ii. *Vector Equalities and Inequalities in E^n*

Define in the vector space E^n vectors **x** and **y** where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

Then
1. $\mathbf{x} = \mathbf{y}$ iff $x_j = y_j, j = 1, \ldots, n$.
2. $\mathbf{x} \geq \mathbf{y}$ iff $x_j \geq y_j, j = 1, \ldots, n$.
3. $\mathbf{x} > \mathbf{y}$ iff $x_j > y_j, j = 1, \ldots, n$.

iii. *A Word on Vector Products*

The reader may wonder since we have a well-defined vector addition that associates with any two vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ a vector $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3 \in V$, whether it is possible to define a *vector product* that associates with $\mathbf{v}_1, \mathbf{v}_2 \in V$ a vector $\mathbf{v}_1\mathbf{v}_2 \in V$. The reader may recall, for example, the *cross product* often used in physics, which does this. (The nonassociativity of the cross product is an undesirable feature of that particular "multiplication.") The answer is that this is sometimes possible, and if the multiplication has an appropriate set of properties, the vector space becomes a *linear algebra*, which we mention more to indicate an ambiguity of connotation than to introduce a new concept.

Of greater interest to us here is the "dot product" or "scalar product," which we prefer to define here as *inner product* on the vector space E^n.

Definition. If $\mathbf{v}_1, \mathbf{v}_2 \in E^n$, then the *inner product* of $\mathbf{v}_1, \mathbf{v}_2$, which we shall write $\mathbf{v}_1 \cdot \mathbf{v}_2$, or $\mathbf{v}_1\mathbf{v}_2$ is defined as the scalar

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{i=1}^{n} v_{1i} v_{2i} = \mathbf{v}_1\mathbf{v}_2$$

Example 1.2

$$\begin{pmatrix} -1 \\ 1 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} = 2 + 0 + 0 + 2 = 4$$

iv. *Additional Concepts*

a. LINEAR COMBINATIONS

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \ldots,\}$ be a collection of vectors. Then a sum of the form $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n + \cdots$, where $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are arbitrary scalars, is called a *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n, \ldots$. If the set of vectors is finite, we have a *finite linear combination.*

Theorem 1.1

Let W be a collection of vectors from a vector space V. Let $S(W) =$ the set of all linear combinations of elements of W. Then $S(W)$ is a vector space.

The set $S(W)$ in Theorem 1.1 is called the *subspace spanned by the set W*. Also, W is said to *span* $S(W)$, or to be a *spanning set* for $S(W)$.

Given a vector space, there may be an infinite number of spanning sets. For example, in E^3 the line L in Figure 1.1 is a subspace of E^3, and it is spanned by any point (vector) other than (0, 0, 0) on the line.

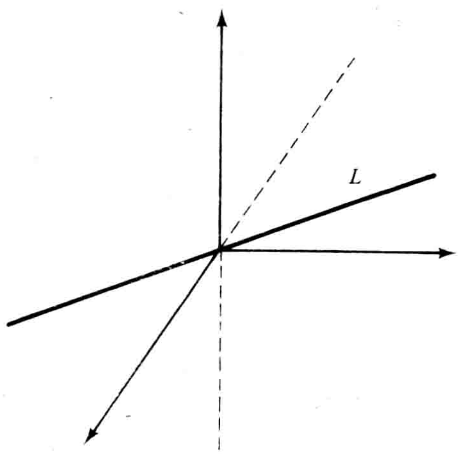


Figure 1.1

b. LINEAR DEPENDENCE AND INDEPENDENCE

Of particular importance is the concept of a linearly independent set of vectors.

Definition. Let V be a vector space in E^k. Let W be a subset of V with $W = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. W is said to be a *linearly independent* set of vectors if

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0}$$

implies that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

That is, the vectors are linearly independent if the *only* way a linear combination of them can equal the zero vector, or *null vector*, of V is for each coefficient of the linear combination to be zero.

Definition. A set of vectors that is not linearly independent is said to be *linearly dependent*.

Several remarks follow almost immediately from these definitions and should be verified by the student.

1. Any set of vectors from a vector space V that contains the null vector is linearly dependent.
2. Every subset of a linearly independent set is linearly independent.
3. A set consisting of a single nonnull vector is linearly independent.
4. If $S_1 \subseteq S_2$ and S_1 is linearly dependent, so is S_2.

The notion of linear independence is intimately related to the ability to express one element of a set of vectors as a linear combination of the others.

For example, consider a set of linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then we have

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0} \tag{1.1}$$

and $\lambda_r \neq 0$ for some integer r. Rewriting (1.1), we have then

$$\lambda_r\mathbf{v}_r = -\lambda_1\mathbf{v}_1 - \lambda_2\mathbf{v}_2 - \cdots - \lambda_{r-1}\mathbf{v}_{r-1} - \lambda_{r+1}\mathbf{v}_{r+1} - \cdots - \lambda_n\mathbf{v}_n \tag{1.2}$$

Since $\lambda_r \neq 0$, we may divide by it in (1.2), obtaining

$$\mathbf{v}_r = -\frac{\lambda_1}{\lambda_r}(\mathbf{v}_1) - \frac{\lambda_2}{\lambda_r}(\mathbf{v}_2) - \cdots - \frac{\lambda_{r-1}}{\lambda_r}(\mathbf{v}_{r-1}) - \frac{\lambda_{r+1}}{\lambda_r}(\mathbf{v}_{r+1}) - \cdots - \frac{\lambda_n}{\lambda_r}(\mathbf{v}_n)$$

Conversely, if $\mathbf{v}_r$ may be so expressed, for some r, then the set is linearly dependent; for if we have

$$\mathbf{v}_r = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{r-1}\mathbf{v}_{r-1} + c_{r+1}\mathbf{v}_{r+1} + \cdots + c_n\mathbf{v}_n$$

then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{r-1}\mathbf{v}_{r-1} - 1\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

and not all the coefficients are zero.