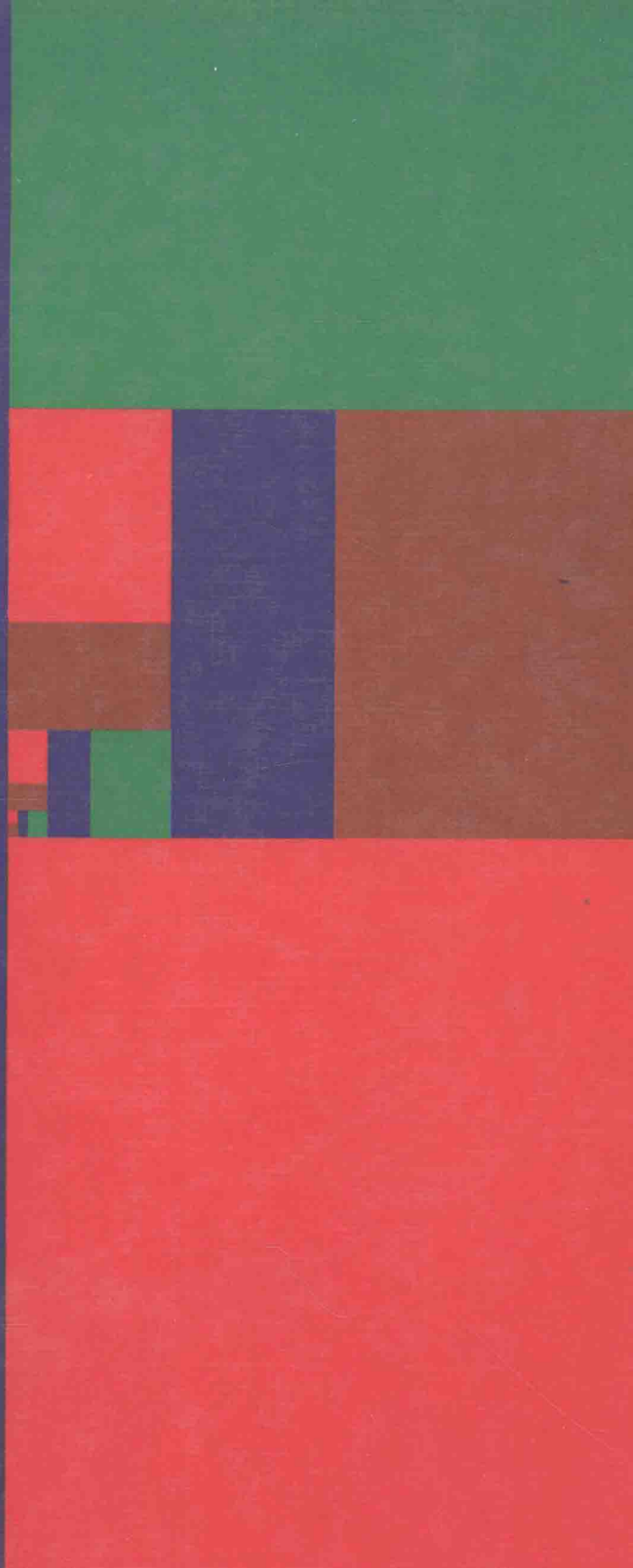
Modern Mathematics
An Elementary Approach

6th Edition

Ruric E. Wheeler



MATHEMATICS

An Elementary Approach

SIXTH EDITION

Ruric E. Wheeler

Samford University

Brooks/Cole Publishing Company
Monterey, California

CONTEMPORARY UNDERGRADUATE MATHEMATICS SERIES
Robert J. Wisner, Consulting Editor

Brooks/Cole Publishing Company
A Division of Wadsworth, Inc.

Printed in the United States of America
10 9 8 7 6 5 4 3 2

Library of Congress Cataloging in Publication Data

Wheeler, Ruric E. [date]
Modern mathematics.

(Contemporary undergraduate mathematics series)
Includes index.
1. Mathematics—1961– . I. Title. II. Series.
QA39.2.W46 1984 510 83-26163
ISBN 0-534-02843-8

Acquisition Editor: *Craig Barth*
Production Service: *Greg Hubit Bookworks, Larkspur, California*
Production Coordinator: *Joan Marsh*
Text Design: *John Edeen*
Cover Design: *Rudolph de Harak, Rudolph de Harak & Associates, Inc.*
Illustrations: *Ayxa Art (technical), Drake Jordan ("Just for Fun")*
Typesetting: *Syntax International*

Photo Credits

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Preface

To The Instructor

One would expect in preparing a sixth edition for a widely used textbook that little revision would be necessary. However, such has never been true of this book. Changes are always needed to accommodate our improved perception of the needs of students. A recent reviewer commented that revisions have been so comprehensive that a comparison of the present book with earlier editions would suggest a completely new book written by a different author.

Every change in the present edition was included to satisfy one of six major goals:

1. To make problem solving more understandable for students, with particular emphasis upon the needs of the student with below average mathematics background.
2. To present elementary mathematics from an adult point of view, emphasizing the understanding of concepts.
3. To give increased emphasis to the ten basic skills as outlined by NCTM.
4. To prove to students that mathematics is not "just calculations" but is a fun, beautifully organized, artistic endeavor.
5. To provide maximum flexibility to the teacher in the selection of topics and methods of presentation.
6. To communicate more closely with students using the book.

The new material of the fifth edition was problem solving, which became a thread interwoven through the book not only to tie together related topics but to work toward the NCTM goal of helping students become effective solvers of problems. All goals were reached with one exception. The author noted with regret that some average and most below average students had no greater appreciation for or interest in problem solving than in other aspects of mathematics. These students simply memorized enough to pass their courses.

To assist students in developing a greater appreciation of problem-solving techniques, the author begins this edition with some of the techniques of cognitive reasoning that have been developed by psychologists to help students improve

intelligence test scores. In the first two sections (and in the appendix) are found some of the cognitive reasoning techniques that not only should improve the response of students to problem solving, but should also improve the attitude of the student toward mathematics.

This edition continues to maintain the same flavor of a mathematical language and logical approach as has characterized the first five editions. Through five editions, this book has emphasized these ten basic skills: computational skills; reasonableness of results; approximation and estimation; using mathematics to solve everyday problems; geometric skills; measurement; constructing and interpreting tables, charts and graphs; recognizing the reasonableness of statistical measurements; understanding the calculator and the computer; and using mathematics to predict or make decisions. Much effort has been expended to sharpen these skills in this edition.

Fifteen years of use in formal college courses have indicated the effectiveness of the book's style—simplicity of language, numerous examples, and intuitive discussions—in assisting the student to enjoy and appreciate mathematics.

The material in this book is presented so as to offer instructors maximum flexibility in designing their courses. Only four sections are essential in Chapter 1, three in Chapter 2, five in Chapter 3, two in Chapter 4, and none in Chapters 6 and 7. Thus, the teacher can select topics to attain the desired emphasis. Likewise, instead of the usual "explain, then apply" approach, the teacher can use the wide variety of problems in the text together with laboratory activities in the supplemental student manual to vary day-by-day teaching procedures.

The sixth major goal of increasing communication with students is accomplished by two very noticeable changes in this edition. The author has written a personal note to the student at the beginning of each chapter and at the beginning of each section of a chapter. Not only is the content of each section described in nonmathematical language, but an attempt is made to indicate where the material of the section will be used in the remainder of the book. To substantiate the problem-solving thread, a problem is described in the special note at the beginning of each chapter—a problem that the student should be able to solve using the material of the chapter. The problem is always solved at the end of the chapter.

In addition, a very comprehensive *Student Self-Study and Laboratory Activities Manual* is available. The student can use it to explore the thinking processes used in both discovering and using mathematics. For each section in the textbook, the *Manual* contains

1. Learning goals or objectives,
2. Numerous laboratory activities,
3. Problems designed for solution on a calculator,
4. Pages from elementary school textbooks.

For each chapter of the book, the *Manual* contains

1. A summary of the chapter,
2. Mistakes commonly made,

3. A practice test,
4. Solutions or partial solutions for selected "Just for Fun" problems.

Over 300 mathematics teachers assisted the author in deciding on relevant material and in the arrangement of the topics. For example, here are responses to a few of the many topics on the questionnaire.

1. Beginning the book with material on cognitive thinking skills in order to quickly develop maturity for problem solving. (The percentage that highly approved, approved, were neutral, or disapproved were 31%, 35%, 27%, 7%, respectively.)
2. Increasing geometry coverage by 20% (41%, 27%, 31%, 1%).
3. Increasing the material on algebra by 15% (26%, 35%, 32%, 7%).
4. Additional material on decimals (16%, 45%, 36%, 3%).
5. Combining the operations for whole numbers with the computational algorithms for these operations (26%, 42%, 21%, 11%).
6. Moving the discussion of metric units to the chapter on measurement (30%, 53%, 13%, 4%).
7. The inclusion of a large number of simple illustrations and discussions while maintaining the same mathematical standards of expecting an understanding of concepts (35%, 37%, 18%, 10%).
8. Introducing each chapter with a problem that is solved using problem-solving techniques and maintaining the thread of problem solving through the 14 chapters (37%, 36%, 20%, 7%).

Each exercise set starts with easy problems. After an asterisk (*) the problems become somewhat harder, and after a double asterisk (**) you will find the challenges for "A" students. That's right. Plenty of problems are included for everyone.

An Instructor's Manual is available from the publisher; it contains suggestions for presenting the material, readiness tests, multiple-choice tests, and answers to all problems not answered in this book.

I am most grateful to the following mathematicians who assisted not only in the sixth edition but in previous editions as well. Through letters and responses to questionnaires, these friends have been most helpful.

Alabama — Jean A. Blake, Alabama A & M University; Dwight Holcomb, Snead State Junior College; Mary Hudson, Samford University; Walter G. McQueen, Chattahoochee Valley Community College; Winston Moody, Alabama A & M University; W. D. Peeples, Jr., Samford University; Shirley A. Wilson, Auburn University at Montgomery.

California — M. A. Fitting, San Jose State University; Richard H. Strand, Azusa Pacific University.

Colorado — Clifford Britton, Mesa College.

Florida	W. Kosmala, University of Tampa; Charles C. Miles, Hillsborough Community College.
Georgia	Richard Cowan, Shorter College; J. Norman Wells, Georgia Southern College.
Hawaii	Richard K. Coburn, Brigham Young University.
Idaho	Alan Clark, Steven Terry, Ricks College.
Illinois	P. J. Blus, National College of Education; Benny F. Tucker, Illinois State University.
Indiana	Sister Florence Marie Rose, Marian College.
Iowa	Charles M. Lindsay, Coe College.
Kansas	Joan Pearson, Labette Community College.
Kentucky	Stanley E. Baker, Sue Bennett College; Clevis D. Carter, Union College.
Louisiana	V. J. DuRapau, Jr., Xavier University; Barbara C. Jones, Northwest Louisiana University.
Maine	George Langbehn, Anthony Soychak, University of Southern Maine.
Maryland	Don Cathcart, Salisbury State College.
Massachusetts	John Tobey, North Shore Community College.
Minnesota	Francis T. Hannick, Mankato State University; Sandy Olson, Winona State University; William Tomhave, University of Minnesota, Morris.
Mississippi	Reba H. Davis, Northeast Mississippi Junior College; Billy R. Sneed, Itawamba Junior College.
Missouri	Glen Haddock, University of Missouri, Rolla; Jerry Wilkerson, Missouri Western State College.
Nebraska	Robert Colling, Kearney State College; Frank F. Ferguson, Chadron State College; Chris Masters, Doane College.
New Hampshire	Richard Evans, Plymouth State College.
New Jersey	Esther E. Guerin, Seton Hall University.
New York	Stephen F. West, State University of New York.
North Carolina	Emmett S. Sams, Mars Hill College; Jerry D. Taylor, Campbell University.
North Dakota	Roger Riveland, Bismarck Junior College.
Ohio	A. P. Malone, Raymond Walters General & Technical College.
Oregon	Don Burdg, Southwestern Oregon Community College; Douglas Cruikshank, Linfield College; Harold N. Hauser, Mt. Hood Community College; Roger B. Nelson, Lewis & Clark College.
Pennsylvania	Cloyd H. Ebersole, Lebanon Valley College; Ephraim B. Fithian, Kutztown State College; Joseph A. Freiwald, Montgomery County Community College; John T. Held, Gettysburg College; Paul K. Horrell, Shippensburg State College; C. Ralph Verno, West Chester State College; Ramon J. Voltz, Grove City College; C. T. Wolf, Millersville State College.
South Carolina	R. A. Lytle, University of South Carolina; Wade H. Sherard, Furman University;

M. B. Ulmer, University of South Carolina at Spartanburg; Bruce F. White, Lander College.

Tennessee — Larry Bouldin, Roane State Community College; Bob Forrester, Volunteer State Community College; Betty Frost, Jackson State Community College; Charles E. Hampton, Free Will Baptist Bible College; Lester C. Hartsell, East Tennessee State University; John Holland, David Lipscomb College; Richard McCord, Middle Tennessee State University; James G. Ware, University of Tennessee; Ernest Woodward, Austin Peay State University.

Texas — Judith Aronow, Lamar University at Orange; James R. Boone, Texas A & M University; Don Brown, Alvin Community College; Robert Netherland, Tarleton State University; Roger Osborn, University of Texas at Austin; Sister M. Geralda Schaefer, Pan American University; Don Skow, Pan American University.

Utah — Bart Nelson, Snow College.

Virginia — Genevieve Knight, Hampton Institute; C. Michael Lohr, Virginia Commonwealth University.

Washington — J. Maurice Kingston, University of Washington.

West Virginia — James R. Keller, Marshall University.

Wisconsin — Ronald D. Dettmers, University of Wisconsin – Whitewater; George L. Henderson, University of Wisconsin – Eau Claire; Duane R. Paull, University of Wisconsin – Eau Claire.

Wyoming — A. Duane Porter, University of Wyoming.

Many mathematics teachers responded to a questionnaire concerning changes that should be made in the sixth edition. I am most appreciative of suggestions made by the following people:

Alabama — Lila F. Acker, Angela Hernandez, W. A. Hopper

Arizona — Jean S. Overholser

Arkansas — Loyde Beam, Sherry Blust, Louis Dunlap, Dorothy Zarelli

California — Alan W. Holz, James R. Ledford, Professor McLean, Eddie M. Norvell, Spiro J. Politis

Colorado — Arlys W. Hanson

Florida — Patricia Mathis, H. E. Payne, Jr., Barry A. Tosh

Georgia — Fred Maynard

Illinois — John A. Dossey, Neal McKenna, Katherine Pedersen, Richard J. Wilders

Indiana — Don S. Balka

Iowa — Merlin Fischer, Sheryl Griffith, Ed Rathmell

Kansas — Forrest Coltharp, Helen Kriegsman, Annabelle Loy, Sister Catherine Therese

Kentucky — Harvey L. Elder

Louisiana — Judy Bergman, Brenda Broussard, Charles Hancock, Ray Robicheaux

Maryland	Robert L. Carson, Robert J. Garner, John P. Jones, Connie Seaman-Cookson, Don Shriner
Massachusetts	Arnold F. Checchi, Joanne Collamore, Carol Iturralde, Margaret J. Kenney, Clifford Pope
Michigan	Frank C. Sherburne
Mississippi	Bettie S. Akers, Sue Key, Nellie Tindall
Missouri	Ronald F. Eldringhoff
Montana	M. B. Banning, Jim Trudnowski
Nebraska	Charles G. Pickens
New Jersey	Professor McCullough
New Mexico	James L. Evans, Russell W. Grover
New York	Beth Reynolds, Ray M. Snider, Henry Snyder
North Carolina	Julian Aydlett, Grace P. Foster, Richard C. Haworth, Gladys M. Kerr, Judy McInerney, Joyce E. Speas
Ohio	Ronald Evans, J. L. Kilpatrick, Richard Rappach, Sally Ann Sikorski
Oregon	James Barnard, Max Henry, Vern Hiebert, Ron Waife
Pennsylvania	James Comella, Robert Dolan
South Carolina	Emma Owens, Charles M. Walker
South Dakota	Verlyn L. Lindell, Betty Monsees, Jan Vandever
Tennessee	Robert Simpson, Wanda Vickery
Texas	Gene Evans, B. G. Nunley, Debbie Pace, Rose Marie Smith, Alfred E. Springer, Ara B. Sullenberger
Utah	Robert Heal, Gail P. Miles, David C. Tolman
Vermont	Sister Katherine O'Donnell
Virginia	Jane Randall
Washington	Gary D. Peterson
West Virginia	Sandra Gum, Al Johnson
Wisconsin	Barbara L. Boe, Sister Judith Brower, Harald M. Neiss

The author expresses sincere appreciation to the reviewers who offered valuable suggestions for this revision: Betty Freemal, University of Cincinnati; Robert Fullerton, Palomar College; George Henderson, University of Wisconsin; Irving Hollingshead, Kutztown State College; Alan Holz, California State Polytechnic University, San Luis Obispo; John Huber, Pan American University; Larry Johnson, Central Missouri State University; Linda Oldaker, Arizona State University; Wayne Rich, Utah State University; James Simpson, University of Kentucky; Joe K. Smith, North Kentucky University; V. Merriline Smith, California State Polytechnic University, Pomona; and Edward C. Wallace, Pan American University.

Thank you, Bob Wisner, for reading the material for the sixth time and for

your many valuable suggestions. Thank you, Cathie Mitchell and Ed Wheeler, for your excellent critique of the material. Credit is also due Greg Hubit Bookworks and Craig Barth and the Brooks/Cole staff for excellent workmanship. The author is grateful for the assistance given by Debbie Driver and Karen Jones. Finally, the author would like to thank Debra Lincoln for patience and understanding while typing the manuscript.

To The Student

At the beginning of each chapter and at the beginning of each section of a chapter, I shall be writing notes to you. In these notes I shall be accomplishing a variety of things. I shall describe in nonmathematical language the content of the section. Often I shall indicate where the present material is used elsewhere in the book; occasionally I shall share some historical perspective or special application of the material at hand. In general, I shall be helping place the specifics of what you are learning in the larger context of the goals of the course.

In addition, an activities and self-study manual has been prepared for you. (Read about this student manual in the note to the teacher.) You should be able to read and comprehend this manual with a minimum of assistance from your teacher. This manual is written to help you learn and **enjoy** elementary mathematics, regardless of your background.

As you begin this study, it would be appropriate to answer the question, "What is mathematics?" Some will say that mathematics consists of operations with numbers used to answer the questions "How many?" and "How much?" In this role, mathematics is a **tool,** a collection of skills that may be used for calculation and problem solving. As a tool, mathematics is indispensable in our modern world of business transactions, industrial production, and scientific research. Because of this, mathematics becomes important to you. To be a competent citizen, you must be able to cope with taxes, interest, budgets, grocery bills, and a never-ending flow of statistics, all of which depend on mathematics. Yet mathematics is more than a tool.

To some of your friends, mathematics may be a science. This answer is also correct. Mathematics is one of the purer sciences. As a science of logical reasoning, the study of mathematics involves a study of methods for drawing conclusions from assumed premises. Some of the greatest minds of every age have contributed to the systematized mathematical knowledge in use today.

The description of mathematics that is appropriate to everyone, however, is that mathematics **is a language.** It is the language of those who wish to express ideas of size, shape, quantity, or order. Thus, to communicate effectively in the marketplace, in the classroom, in public affairs, and even at home, it is essential that one have a knowledge of this language.

The one thing that mathematics is not is a spectator sport. To enjoy mathematics, you must become involved; so my primary goal in this book is to help you get in the habit of using the language of mathematics and to help you get involved as a mathematical problem solver. Remember, you are the one who will benefit if you learn well how to use the language of mathematics and to become a proficient problem solver. Good luck.

Ruric E. Wheeler

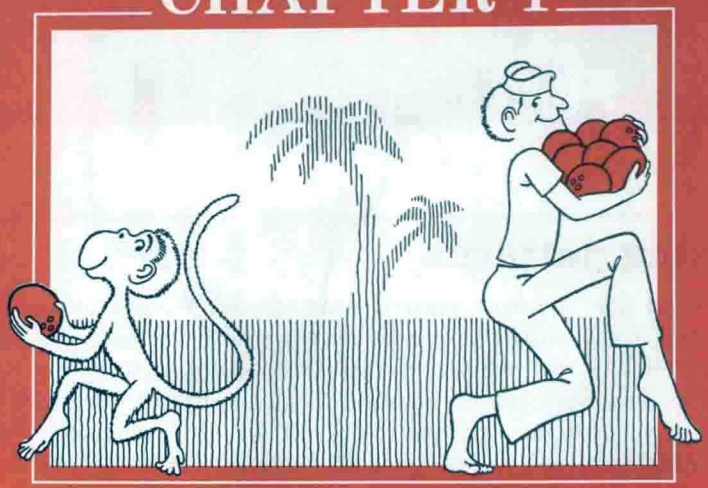
Course Outlines

	Students with an average mathematical background (or less)	Better-prepared students
One-semester courses		
Courses for liberal arts majors	Sections 1–4 of Chapter 1 Sections 1–3, 5, 6 of Chapter 2 Sections 1–3 of Chapter 3 Sections 1–3 of Chapter 4 Chapter 5 Sections 1, 2 of Chapter 6 Sections 1, 3, 4 of Chapter 7 Sections 1–3 of Chapter 13 Sections 1–3 of Chapter 14	Sections 1–5 of Chapter 1 Sections 1–3, 5–7 of Chapter 2 Sections 1–4 of Chapter 3 Sections 1–3, 6 of Chapter 4 Sections 1–5 of Chapter 6 Sections 1, 3, 4 of Chapter 7 Sections 1–4 of Chapter 12 Sections 1–3 of Chapter 13 Sections 1–4 of Chapter 14
Courses for elementary education majors	Sections 1–4 of Chapter 1 Sections 1–3 of Chapter 2 Sections 1–6 of Chapter 3 Sections 1–3 of Chapter 4 Chapter 5 Sections 1–3, 5, 6 of Chapter 8 Sections 1, 2, 4, 5 of Chapter 9	Sections 1–4 of Chapter 1 Sections 1–3 of Chapter 2 Sections 1–6 of Chapter 3 Sections 1–3, 6 of Chapter 4 Chapter 5 Sections 1–4 of Chapter 6 Sections 1–3, 5, 6 of Chapter 8 Sections 1, 2, 4, 5 of Chapter 9
Combination courses for liberal arts majors and elementary education majors	Sections 1–4 of Chapter 1 Sections 1–3, 5, 6 of Chapter 2 Sections 1–6 of Chapter 3 Sections 1–3 of Chapter 4 Chapter 5 Sections 1–3, 5, 6 of Chapter 8 Sections 1–3 of Chapter 14	Sections 1–4 of Chapter 1 Sections 1–3, 5, 6 of Chapter 2 Sections 1–6 of Chapter 3 Sections 1–3, 6 of Chapter 4 Chapter 5 Sections 1–3, 5, 6 of Chapter 8 Sections 1–2, 4, 5 of Chapter 9 Sections 1–4 of Chapter 14
Two-semester courses		
Courses for liberal arts majors	Sections 1–4 of Chapter 1 Chapter 2 Sections 1–4 of Chapter 3 Sections 1–3, 6 of Chapter 4 Chapters 5, 6, 7, 8, 9 Sections 1–5 of Chapter 11 Sections 1–5 of Chapter 12 Chapters 13, 14	Chapters 1, 2 Sections 1–4 of Chapter 3 Chapters 4, 5, 6, 7, 8, 9, 11, 12, 13, 14
Courses for elementary education majors	Sections 1–4 of Chapter 1 Sections 1–4 of Chapter 2 Sections 1–6 of Chapter 3 Sections 1–3, 6 of Chapter 4 Chapters 5, 6 Sections 1, 3, 4 of Chapter 7 Chapters 8, 9 Sections 1–3 of Chapter 10 Sections 1–5 of Chapter 11 Chapters 13, 14	Chapter 1 Sections 1–4 of Chapter 2 Chapter 3 Sections 1–3, 6 of Chapter 4 Chapters 5, 6 Sections 1–3, 4 of Chapter 7 Chapters 8, 9, 10 Sections 1–5 of Chapter 11 Chapters 13, 14
Combination courses for liberal arts majors and elementary education majors	Sections 1–4 of Chapter 1 Chapter 2 Sections 1–6 of Chapter 3 Sections 1–3, 6 of Chapter 4 Chapters 5, 6 Sections 1, 3, 4 of Chapter 7 Chapters 8, 9, 10, 11, 13, 14	Chapters 1, 2, 3 Sections 1–3, 6 of Chapter 4 Chapters 5, 6, 7, 8, 9, 10, 11, 13, 14

Contents

* Denotes sections that may be omitted without loss of continuity or comprehension.

CHAPTER 1



Cognitive Reasoning and Problem Solving

Do you want to be able to analyze and comprehend complex descriptions and directions? Do you want to improve your problem-solving ability? Do you want to improve your test scores on true-false, multiple-choice, and standardized tests such as the GRE, the NTE, and the LSAT? This chapter will introduce to you some aids to productive thinking, sometimes called *cognitive reasoning*. The techniques of cognitive reasoning should be helpful in at least four areas—on tests, in school, on the job, and in everyday life.

Furthermore, we have found that some of the concepts of cognitive reasoning are very helpful in bridging the gap between high school mathematics (that you may have had many years ago) and beginning college mathematics. This first chapter is written to help you learn, and we hope enjoy, elementary mathematics, regardless of your background. Also, this chapter will help you to become a proficient problem solver. With a concerted effort, you will be able to solve problems like the following when you finish this chapter.

■ Three sailors and a monkey are shipwrecked on a deserted island, where they discover that coconuts are the only available food. The men collect coconuts from all the trees, but because the sun is setting, they decide to wait until morning to divide the supply. However, once Bob and Carl are asleep, Al arises, divides the coconuts into three equal shares, gives the remaining coconut to the monkey, hides his share, restacks the others, and returns to sleep. Later, Bob sneaks out of bed, divides the pile of coconuts into three equal shares, has one nut left over, gives it to the monkey, hides his pile, replaces the remaining coconuts, and returns to sleep. Toward morning, Carl, too, carefully moves from his bed, divides the pile into three

equal shares, has one coconut left over, gives it to the monkey, hides his pile, replaces the remaining coconuts, and returns to bed. After this busy night, the sailors meet again and divide the pile of coconuts. Again, one coconut is left over for the monkey. What is the least possible number of coconuts in the original pile? (See the solution on p. 36.)

1 Discovering patterns

What does the word *pattern* mean to you? We consider a pattern to be any kind of regularity that can be recognized by the mind. You will see many different kinds of patterns in this introductory section. You will first learn to recognize very simple nonmathematical patterns; then you will study some number patterns; and finally you will learn to recognize patterns that will be valuable to you as you gain mathematical maturity and become a problem solver.

Begin your study of patterns by looking at two problems that you might have encountered in an elementary school textbook.

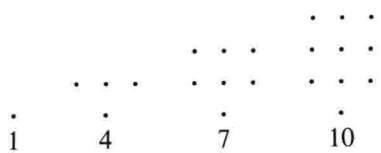
Example • Many tests as well as elementary textbooks relate patterns to input-output machines. We are told that for the given input we get the following output.

What would the machine give as output for the following input?

Select your answer from the following.

Solution Did you observe that the machine inverted (turned upside down) the input image? Hence the answer is (e). •

Example • The dot printer is also often introduced in a study of patterns. A dot *printer is* programmed to print the following designs.



What number will the next pattern represent?

Solution Did you add another row of three dots to the design? The answer is 13. •

Patterns occur in many forms such as the sequence of letters in the alphabet shown in the following example.

Example • Suppose you were asked to write the next two letters that would appear in the following list.

$$B, A, E, D, H, G, K, J, \_\_\_, \_\_\_$$

Would your answer be P, O or S, R or N, M or N, O, or would you say "I don't know" because you were unable to find an answer?

Solution In order to discover patterns, many students need to develop the ability to concentrate on the meaning of each part of the problem. To emphasize the meaning of each part, let's describe what we discover.

First note that A, which follows B in the series, is the letter preceding B in the alphabet. This is a possible pattern. Are there any more pairs of letters with this pattern as a repetition? Are we in luck? How about E, D; H, G; and K, J? Thus, the two letters we are trying to discover must be such that the second letter precedes the first letter in the alphabet. What are the letters? Maybe we need some additional information.

Since the letters evidently go in pairs, let's look at the first letter of each pair. Study carefully the relationship between B and E. What do you note? There are two letters in the alphabet between B and E. Could this be the beginning of a pattern? Yes, there are two letters between E and H, and two between H and K. If there are two between K and our first unknown letter, this letter will be N. From our first discovery, the next letter will be M. Thus, from our list of possible answers, N, M was correct. •

Example • Which set of letters is different from the other three sets?

(a) $BLNA$ (b) $SBDR$ (c) $UFHT$ (d) $CJRD$ (e) All are alike.

Solution Notice that in the first three sets of letters the fourth letter precedes the first letter in the set. For example, in $BLNA$ the fourth letter, A, precedes the first letter, B, in the alphabet. This relationship does not occur in (d). Let's see if there is any other observable pattern in the first three that does not occur in (d). The two middle letters L and N, B and D, and F and H are separated by one letter. In (d), J and R are separated by more than one letter. Therefore, the different set is (d). •