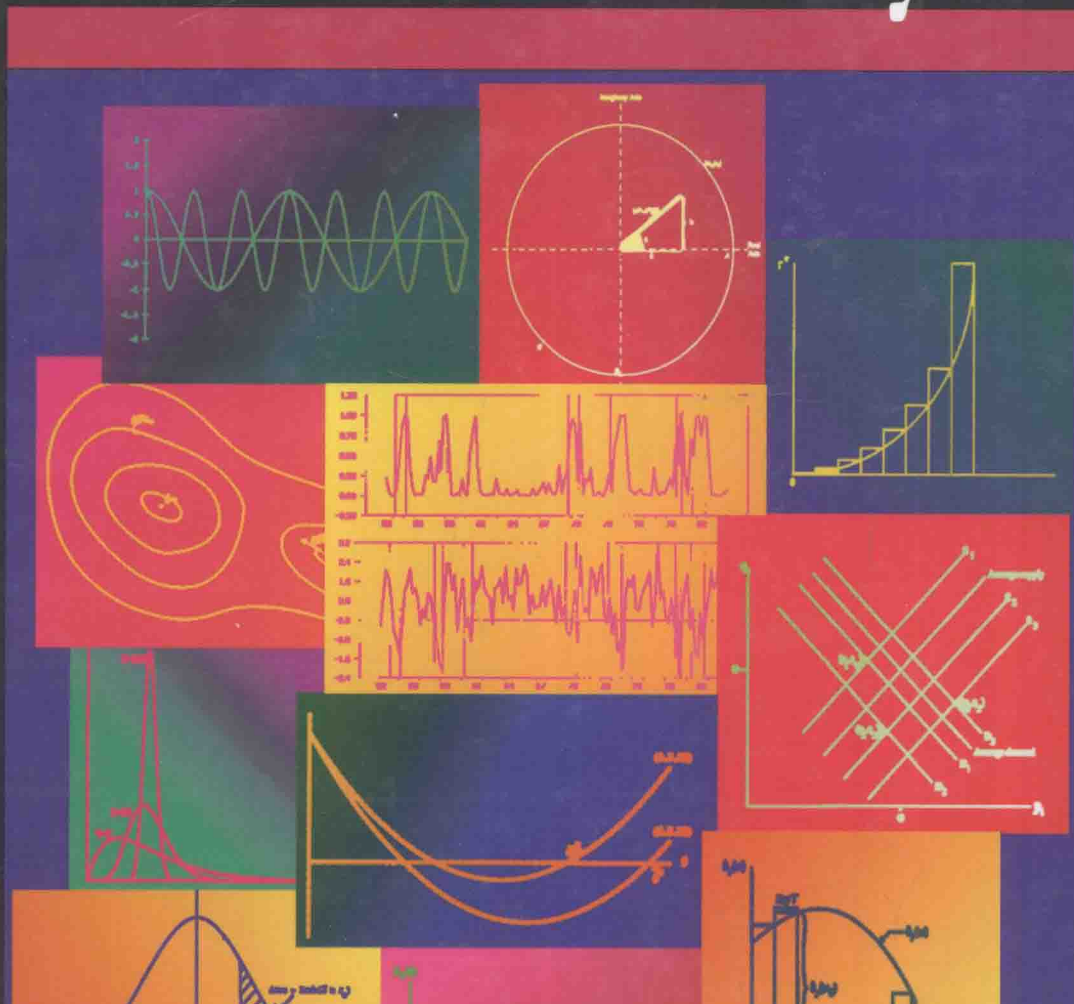
Time Series Analysis



James D. Hamilton

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Preface

Much of economics is concerned with modeling dynamics. There has been an explosion of research in this area in the last decade, as "time series econometrics" has practically come to be synonymous with "empirical macroeconomics."

Several texts provide good coverage of the advances in the economic analysis of dynamic systems, while others summarize the earlier literature on statistical inference for time series data. There seemed a use for a text that could integrate the theoretical and empirical issues as well as incorporate the many advances of the last decade, such as the analysis of vector autoregressions, estimation by generalized method of moments, and statistical inference for nonstationary data. This is the goal of *Time Series Analysis*.

A principal anticipated use of the book would be as a textbook for a graduate econometrics course in time series analysis. The book aims for maximum flexibility through what might be described as an integrated modular structure. As an example of this, the first three sections of Chapter 13 on the Kalman filter could be covered right after Chapter 4, if desired. Alternatively, Chapter 13 could be skipped altogether without loss of comprehension. Despite this flexibility, state-space ideas are fully integrated into the text beginning with Chapter 1, where a state-space representation is used (without any jargon or formalism) to introduce the key results concerning difference equations. Thus, when the reader encounters the formal development of the state-space framework and the Kalman filter in Chapter 13, the notation and key ideas should already be quite familiar.

Spectral analysis (Chapter 6) is another topic that could be covered at a point of the reader's choosing or skipped altogether. In this case, the integrated modular structure is achieved by the early introduction and use of autocovariance-generating functions and filters. Wherever possible, results are described in terms of these rather than the spectrum.

Although the book is designed with an econometrics course in time series methods in mind, the book should be useful for several other purposes. It is completely self-contained, starting from basic principles accessible to first-year graduate students and including an extensive math review appendix. Thus the book would be quite suitable for a first-year graduate course in macroeconomics or dynamic methods that has no econometric content. Such a course might use Chapters 1 and 2, Sections 3.1 through 3.5, and Sections 4.1 and 4.2.

Yet another intended use for the book would be in a conventional econometrics course without an explicit time series focus. The popular econometrics texts do not have much discussion of such topics as numerical methods; asymptotic results for serially dependent, heterogeneously distributed observations; estimation of models with distributed lags; autocorrelation- and heteroskedasticity-consistent standard errors; Bayesian analysis; or generalized method of moments. All of these topics receive extensive treatment in *Time Series Analysis*. Thus, an econometrics course without an explicit focus on time series might make use of Sections 3.1 through 3.5, Chapters 7 through 9, and Chapter 14, and perhaps any of Chapters 5, 11, and 12 as well. Again, the text is self-contained, with a fairly complete discussion of conventional simultaneous equations methods in Chapter 9. Indeed, a very important goal of the text is to develop the parallels between (1) the traditional econometric approach to simultaneous equations and (2) the current popularity of vector autoregressions and generalized method of moments estimation.

Finally, the book attempts to provide a rigorous motivation for the methods and yet still be accessible for researchers with purely applied interests. This is achieved by relegation of many details to mathematical appendixes at the ends of chapters, and by inclusion of numerous examples that illustrate exactly how the theoretical results are used and applied in practice.

The book developed out of my lectures at the University of Virginia. I am grateful first and foremost to my many students over the years whose questions and comments have shaped the course of the manuscript. I also have an enormous debt to numerous colleagues who have kindly offered many useful suggestions, and would like to thank in particular Donald W. K. Andrews, Jushan Bai, Peter Bearse, Stephen R. Blough, John Cochrane, George Davis, Michael Dotsey, John Elder, Robert Engle, T. Wake Epps, Marjorie Flavin, John Geweke, Eric Ghysels, Carlo Giannini, Clive W. J. Granger, Alastair Hall, Bruce E. Hansen, Kevin Hassett, Tomoo Inoue, Ravi Jagannathan, Kenneth F. Kroner, Jaime Marquez, Rocco Mosconi, Edward Nelson, Masao Ogaki, Adrian Pagan, Peter C. B. Phillips, Peter Rappoport, Glenn Rudebusch, Raul Susmel, Mark Watson, Kenneth D. West, Halbert White, and Jeffrey M. Wooldridge. I would also like to thank Pok-sang Lam and John Rogers for graciously sharing their data. Thanks also go to Keith Sill and Christopher Stomberg for assistance with the figures, to Rita Chen for assistance with the statistical tables in Appendix B, and to Richard Mickey for a superb job of copy editing.

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1 Difference Equations

1.1. *First-Order Difference Equations*

This book is concerned with the dynamic consequences of events over time. Let's say we are studying a variable whose value at date t is denoted y_t. Suppose we are given a dynamic equation relating the value y takes on at date t to another variable w_t and to the value y took on in the previous period:

$$y_t = \phi y_{t-1} + w_t. \qquad [1.1.1]$$

Equation [1.1.1] is a *linear first-order difference equation*. A *difference equation* is an expression relating a variable y_t to its previous values. This is a *first-order* difference equation because only the first lag of the variable (y_{t-1}) appears in the equation. Note that it expresses y_t as a linear function of y_{t-1} and w_t.

An example of [1.1.1] is Goldfeld's (1973) estimated money demand function for the United States. Goldfeld's model related the log of the real money holdings of the public (m_t) to the log of aggregate real income (I_t), the log of the interest rate on bank accounts (r_{bt}), and the log of the interest rate on commercial paper (r_{ct}):

$$m_t = 0.27 + 0.72m_{t-1} + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}. \qquad [1.1.2]$$

This is a special case of [1.1.1] with $y_t = m_t$, $\phi = 0.72$, and

$$w_t = 0.27 + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}.$$

For purposes of analyzing the dynamics of such a system, it simplifies the algebra a little to summarize the effects of all the input variables (I_t, r_{bt}, and r_{ct}) in terms of a scalar w_t as here.

In Chapter 3 the input variable w_t will be regarded as a random variable, and the implications of [1.1.1] for the statistical properties of the output series y_t will be explored. In preparation for this discussion, it is necessary first to understand the mechanics of difference equations. For the discussion in Chapters 1 and 2, the values for the input variable $\{w_1, w_2, \ldots\}$ will simply be regarded as a sequence of deterministic numbers. Our goal is to answer the following question: If a dynamic system is described by [1.1.1], what are the effects on y of changes in the value of w?

Solving a Difference Equation by Recursive Substitution

The presumption is that the dynamic equation [1.1.1] governs the behavior of y for all dates t. Thus, for each date we have an equation relating the value of

y for that date to its previous value and the current value of w:

Date	*Equation*	
0	$y_0 = \phi y_{-1} + w_0$	[1.1.3]
1	$y_1 = \phi y_0 + w_1$	[1.1.4]
2	$y_2 = \phi y_1 + w_2$	[1.1.5]
$\vdots$	$\vdots$	
t	$y_t = \phi y_{t-1} + w_t.$	[1.1.6]

If we know the starting value of y for date $t = -1$ and the value of w for dates $t = 0, 1, 2, \ldots$, then it is possible to simulate this dynamic system to find the value of y for any date. For example, if we know the value of y for $t = -1$ and the value of w for $t = 0$, we can calculate the value of y for $t = 0$ directly from [1.1.3]. Given this value of y_0 and the value of w for $t = 1$, we can calculate the value of y for $t = 1$ from [1.1.4]:

$$y_1 = \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1,$$

or

$$y_1 = \phi^2 y_{-1} + \phi w_0 + w_1.$$

Given this value of y_1 and the value of w for $t = 2$, we can calculate the value of y for $t = 2$ from [1.1.5]:

$$y_2 = \phi y_1 + w_2 = \phi(\phi^2 y_{-1} + \phi w_0 + w_1) + w_2,$$

or

$$y_2 = \phi^3 y_{-1} + \phi^2 w_0 + \phi w_1 + w_2.$$

Continuing recursively in this fashion, the value that y takes on at date t can be described as a function of its initial value y_{-1} and the history of w between date 0 and date t:

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \cdots + \phi w_{t-1} + w_t. \qquad [1.1.7]$$

This procedure is known as solving the difference equation [1.1.1] by *recursive substitution.*

Dynamic Multipliers

Note that [1.1.7] expresses y_t as a linear function of the initial value y_{-1} and the historical values of w. This makes it very easy to calculate the effect of w_0 on y_t. If w_0 were to change with y_{-1} and $w_1, w_2, \ldots, w_t$ taken as unaffected, the effect on y_t would be given by

$$\frac{\partial y_t}{\partial w_0} = \phi^t. \qquad [1.1.8]$$

Note that the calculations would be exactly the same if the dynamic simulation were started at date t (taking y_{t-1} as given); then y_{t+j} could be described as a function of y_{t-1} and $w_t, w_{t+1}, \ldots, w_{t+j}$:

$$y_{t+j} = \phi^{j+1}y_{t-1} + \phi^j w_t + \phi^{j-1}w_{t+1} + \phi^{j-2}w_{t+2} + \cdots + \phi w_{t+j-1} + w_{t+j}. \qquad [1.1.9]$$

The effect of w_t on y_{t+j} is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j. \qquad [1.1.10]$$

Thus the *dynamic multiplier* [1.1.10] depends only on j, the length of time separating the disturbance to the input (w_t) and the observed value of the output (y_{t+j}). The multiplier does not depend on t; that is, it does not depend on the dates of the observations themselves. This is true of any linear difference equation.

As an example of calculating a dynamic multiplier, consider again Goldfeld's money demand specification [1.1.2]. Suppose we want to know what will happen to money demand two quarters from now if current income I_t were to increase by one unit today with future income I_{t+1} and I_{t+2} unaffected:

$$\frac{\partial m_{t+2}}{\partial I_t} = \frac{\partial m_{t+2}}{\partial w_t} \times \frac{\partial w_t}{\partial I_t} = \phi^2 \times \frac{\partial w_t}{\partial I_t}.$$

From [1.1.2], a one-unit increase in I_t will increase w_t by 0.19 units, meaning that $\partial w_t/\partial I_t = 0.19$. Since $\phi = 0.72$, we calculate

$$\frac{\partial m_{t+2}}{\partial I_t} = (0.72)^2(0.19) = 0.098.$$

Because I_t is the log of income, an increase in I_t of 0.01 units corresponds to a 1% increase in income. An increase in m_t of $(0.01)\cdot(0.098) \cong 0.001$ corresponds to a 0.1% increase in money holdings. Thus the public would be expected to increase its money holdings by a little less than 0.1% two quarters following a 1% increase in income.

Different values of ϕ in [1.1.1] can produce a variety of dynamic responses of y to w. If $0 < \phi < 1$, the multiplier $\partial y_{t+j}/\partial w_t$ in [1.1.10] decays geometrically toward zero. Panel (a) of Figure 1.1 plots ϕ^j as a function of j for $\phi = 0.8$. If $-1 < \phi < 0$, the multiplier $\partial y_{t+j}/\partial w_t$ will alternate in sign as in panel (b). In this case an increase in w_t will cause y_t to be higher, y_{t+1} to be lower, y_{t+2} to be higher, and so on. Again the absolute value of the effect decays geometrically toward zero. If $\phi > 1$, the dynamic multiplier increases exponentially over time as in panel (c). A given increase in w_t has a larger effect the farther into the future one goes. For $\phi < -1$, the system [1.1.1] exhibits explosive oscillation as in panel (d).

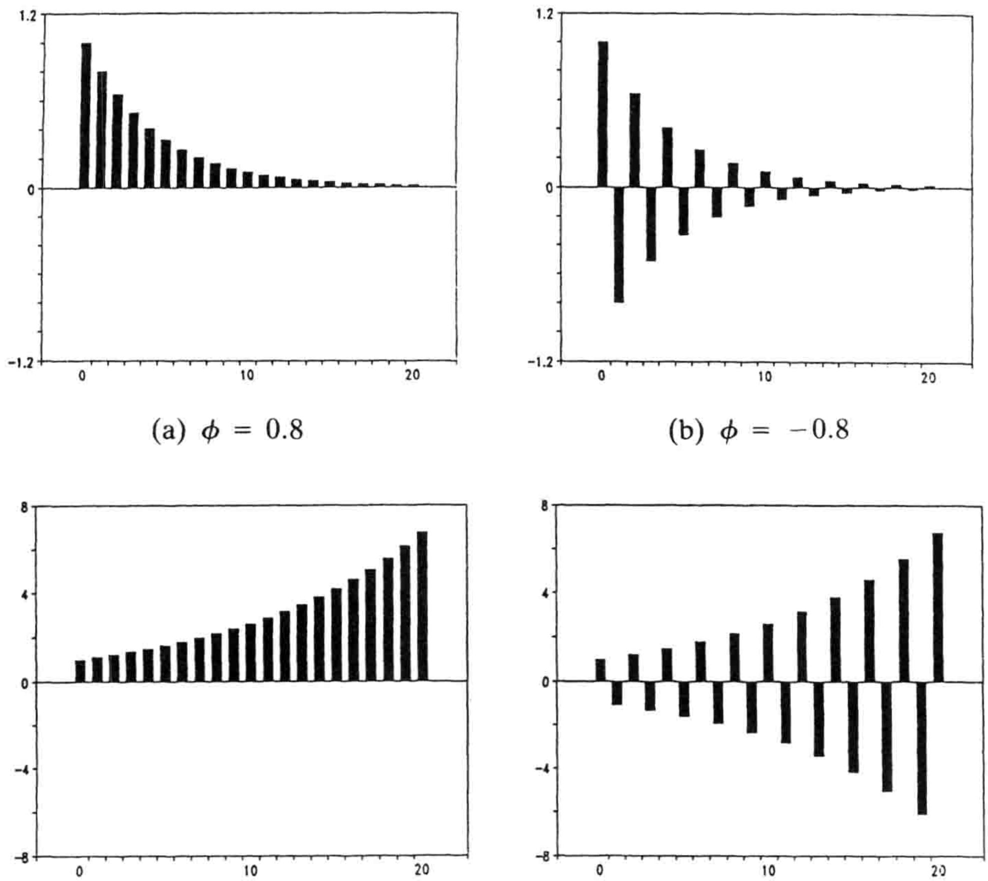
Thus, if $|\phi| < 1$, the system is stable; the consequences of a given change in w_t will eventually die out. If $|\phi| > 1$, the system is explosive. An interesting possibility is the borderline case, $\phi = 1$. In this case, the solution [1.1.9] becomes

$$y_{t+j} = y_{t-1} + w_t + w_{t+1} + w_{t+2} + \cdots + w_{t+j-1} + w_{t+j}. \qquad [1.1.11]$$

Here the output variable y is the sum of the historical inputs w. A one-unit increase in w will cause a permanent one-unit increase in y:

$$\frac{\partial y_{t+j}}{\partial w_t} = 1 \qquad \text{for } j = 0, 1, \ldots .$$

We might also be interested in the effect of w on the present value of the stream of future realizations of y. For a given stream of future values y_t, y_{t+1},



(a) $\phi = 0.8$ (b) $\phi = -0.8$

(c) $\phi = 1.1$ (d) $\phi = -1.1$

FIGURE 1.1 Dynamic multiplier for first-order difference equation for different values of ϕ (plot of $\partial y_{t+j}/\partial w_t = \phi^j$ as a function of the lag j).

$y_{t+2}, \ldots$ and a constant interest rate[1] $r > 0$, the *present value* of the stream at time t is given by

$$y_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \frac{y_{t+3}}{(1+r)^3} + \cdots. \qquad [1.1.12]$$

Let β denote the discount factor:

$$\beta \equiv 1/(1+r).$$

Note that $0 < \beta < 1$. Then the present value [1.1.12] can be written as

$$\sum_{j=0}^{\infty} \beta^j y_{t+j}. \qquad [1.1.13]$$

Consider what would happen if there were a one-unit increase in w_t with $w_{t+1}, w_{t+2}, \ldots$ unaffected. The consequences of this change for the present value of y are found by differentiating [1.1.13] with respect to w_t and then using [1.1.10]

[1]The interest rate is measured here as a fraction of 1; thus $r = 0.1$ corresponds to a 10% interest rate.