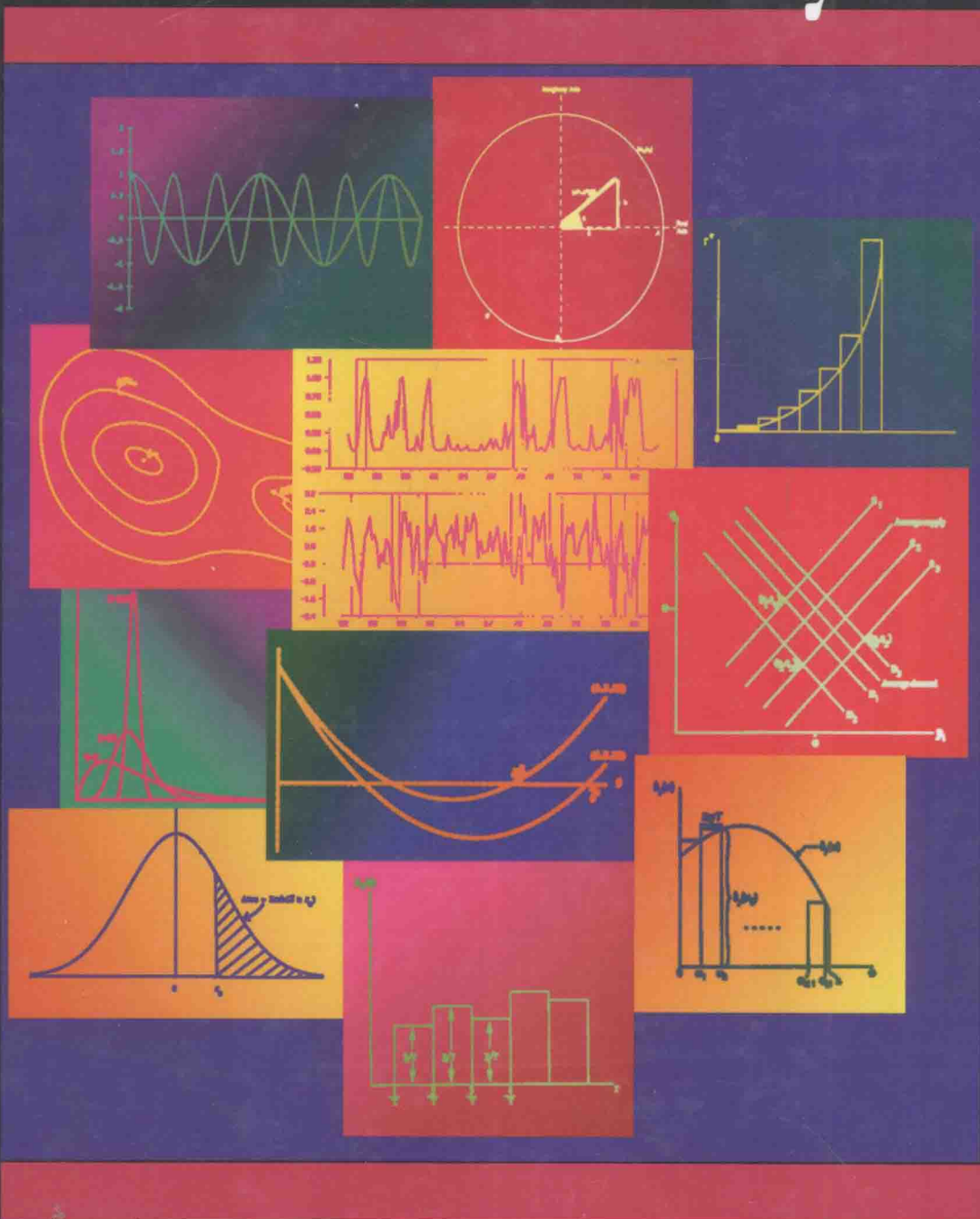


Time Series Analysis



James D. Hamilton

Time Series Analysis

James D. Hamilton

PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY

Copyright © 1994 by Princeton University Press
Published by Princeton University Press, 41 William St.,
Princeton, New Jersey 08540
In the United Kingdom: Princeton University Press,
Chichester, West Sussex

All Rights Reserved

Library of Congress Cataloging-in-Publication Data

Hamilton, James D. (James Douglas), (1954-)

Time series analysis / James D. Hamilton.

p. cm.

Includes bibliographical references and indexes.

ISBN-13: 978-0-691-04289-3 (cloth)

ISBN-10: 0-691-04289-6 (cloth)

1. Time-series analysis. I. Title.

QA280.H264 1994

519.5'5—dc20 93-4958

CIP

This book has been composed in Times Roman.

Princeton University Press books are printed on acid-free paper and meet the guidelines for permanence and durability of the Committee on Production Guidelines for Book Longevity of the Council on Library Resources.

<http://pup.princeton.edu>

Printed in the United States of America

Preface

Much of economics is concerned with modeling dynamics. There has been an explosion of research in this area in the last decade, as “time series econometrics” has practically come to be synonymous with “empirical macroeconomics.”

Several texts provide good coverage of the advances in the economic analysis of dynamic systems, while others summarize the earlier literature on statistical inference for time series data. There seemed a use for a text that could integrate the theoretical and empirical issues as well as incorporate the many advances of the last decade, such as the analysis of vector autoregressions, estimation by generalized method of moments, and statistical inference for nonstationary data. This is the goal of *Time Series Analysis*.

A principal anticipated use of the book would be as a textbook for a graduate econometrics course in time series analysis. The book aims for maximum flexibility through what might be described as an integrated modular structure. As an example of this, the first three sections of Chapter 13 on the Kalman filter could be covered right after Chapter 4, if desired. Alternatively, Chapter 13 could be skipped altogether without loss of comprehension. Despite this flexibility, state-space ideas are fully integrated into the text beginning with Chapter 1, where a state-space representation is used (without any jargon or formalism) to introduce the key results concerning difference equations. Thus, when the reader encounters the formal development of the state-space framework and the Kalman filter in Chapter 13, the notation and key ideas should already be quite familiar.

Spectral analysis (Chapter 6) is another topic that could be covered at a point of the reader’s choosing or skipped altogether. In this case, the integrated modular structure is achieved by the early introduction and use of autocovariance-generating functions and filters. Wherever possible, results are described in terms of these rather than the spectrum.

Although the book is designed with an econometrics course in time series methods in mind, the book should be useful for several other purposes. It is completely self-contained, starting from basic principles accessible to first-year graduate students and including an extensive math review appendix. Thus the book would be quite suitable for a first-year graduate course in macroeconomics or dynamic methods that has no econometric content. Such a course might use Chapters 1 and 2, Sections 3.1 through 3.5, and Sections 4.1 and 4.2.

Yet another intended use for the book would be in a conventional econometrics course without an explicit time series focus. The popular econometrics texts do not have much discussion of such topics as numerical methods; asymptotic results for serially dependent, heterogeneously distributed observations; estimation of models with distributed lags; autocorrelation- and heteroskedasticity-consistent

standard errors; Bayesian analysis; or generalized method of moments. All of these topics receive extensive treatment in *Time Series Analysis*. Thus, an econometrics course without an explicit focus on time series might make use of Sections 3.1 through 3.5, Chapters 7 through 9, and Chapter 14, and perhaps any of Chapters 5, 11, and 12 as well. Again, the text is self-contained, with a fairly complete discussion of conventional simultaneous equations methods in Chapter 9. Indeed, a very important goal of the text is to develop the parallels between (1) the traditional econometric approach to simultaneous equations and (2) the current popularity of vector autoregressions and generalized method of moments estimation.

Finally, the book attempts to provide a rigorous motivation for the methods and yet still be accessible for researchers with purely applied interests. This is achieved by relegation of many details to mathematical appendixes at the ends of chapters, and by inclusion of numerous examples that illustrate exactly how the theoretical results are used and applied in practice.

The book developed out of my lectures at the University of Virginia. I am grateful first and foremost to my many students over the years whose questions and comments have shaped the course of the manuscript. I also have an enormous debt to numerous colleagues who have kindly offered many useful suggestions, and would like to thank in particular Donald W. K. Andrews, Jushan Bai, Peter Bearse, Stephen R. Blough, John Cochrane, George Davis, Michael Dotsey, John Elder, Robert Engle, T. Wake Epps, Marjorie Flavin, John Geweke, Eric Ghysels, Carlo Giannini, Clive W. J. Granger, Alastair Hall, Bruce E. Hansen, Kevin Hassett, Tomoo Inoue, Ravi Jagannathan, Kenneth F. Kroner, Jaime Marquez, Rocco Mosconi, Edward Nelson, Masao Ogaki, Adrian Pagan, Peter C. B. Phillips, Peter Rappoport, Glenn Rudebusch, Raul Susmel, Mark Watson, Kenneth D. West, Halbert White, and Jeffrey M. Wooldridge. I would also like to thank Pok-sang Lam and John Rogers for graciously sharing their data. Thanks also go to Keith Sill and Christopher Stomberg for assistance with the figures, to Rita Chen for assistance with the statistical tables in Appendix B, and to Richard Mickey for a superb job of copy editing.

James D. Hamilton

Contents

PREFACE *xiii*

1 *Difference Equations* **1**

1.1. First-Order Difference Equations 1

1.2. p th-Order Difference Equations 7

APPENDIX 1.A. Proofs of Chapter 1 Propositions 21

References 24

2 *Lag Operators* **25**

2.1. Introduction 25

2.2. First-Order Difference Equations 27

2.3. Second-Order Difference Equations 29

2.4. p th-Order Difference Equations 33

2.5. Initial Conditions and Unbounded Sequences 36

References 42

3 *Stationary ARMA Processes* **43**

3.1. Expectations, Stationarity, and Ergodicity 43

3.2. White Noise 47

3.3. Moving Average Processes 48

3.4. Autoregressive Processes 53

3.5. Mixed Autoregressive Moving Average
Processes 59

- 3.6. The Autocovariance-Generating Function 61
- 3.7. Invertibility 64

APPENDIX 3.A. Convergence Results for Infinite-Order Moving Average Processes 69

Exercises 70 References 71

4 *Forecasting* 72

- 4.1. Principles of Forecasting 72
- 4.2. Forecasts Based on an Infinite Number of Observations 77
- 4.3. Forecasts Based on a Finite Number of Observations 85
- 4.4. The Triangular Factorization of a Positive Definite Symmetric Matrix 87
- 4.5. Updating a Linear Projection 92
- 4.6. Optimal Forecasts for Gaussian Processes 100
- 4.7. Sums of *ARMA* Processes 102
- 4.8. Wold's Decomposition and the Box-Jenkins Modeling Philosophy 108

APPENDIX 4.A. Parallel Between OLS Regression and Linear Projection 113

APPENDIX 4.B. Triangular Factorization of the Covariance Matrix for an *MA(1)* Process 114

Exercises 115 References 116

5 *Maximum Likelihood Estimation* 117

- 5.1. Introduction 117
- 5.2. The Likelihood Function for a Gaussian *AR(1)* Process 118
- 5.3. The Likelihood Function for a Gaussian *AR(p)* Process 123
- 5.4. The Likelihood Function for a Gaussian *MA(1)* Process 127
- 5.5. The Likelihood Function for a Gaussian *MA(q)* Process 130
- 5.6. The Likelihood Function for a Gaussian *ARMA(p, q)* Process 132
- 5.7. Numerical Optimization 133

- 5.8. Statistical Inference with Maximum Likelihood Estimation 142
- 5.9. Inequality Constraints 146

APPENDIX 5.A. Proofs of Chapter 5 Propositions 148
Exercises 150 References 150

6 *Spectral Analysis* 152

- 6.1. The Population Spectrum 152
- 6.2. The Sample Periodogram 158
- 6.3. Estimating the Population Spectrum 163
- 6.4. Uses of Spectral Analysis 167

APPENDIX 6.A. Proofs of Chapter 6 Propositions 172
Exercises 178 References 178

7 *Asymptotic Distribution Theory* 180

- 7.1. Review of Asymptotic Distribution Theory 180
- 7.2. Limit Theorems for Serially Dependent Observations 186

APPENDIX 7.A. Proofs of Chapter 7 Propositions 195
Exercises 198 References 199

8 *Linear Regression Models* 200

- 8.1. Review of Ordinary Least Squares with Deterministic Regressors and i.i.d. Gaussian Disturbances 200
- 8.2. Ordinary Least Squares Under More General Conditions 207
- 8.3. Generalized Least Squares 220

APPENDIX 8.A. Proofs of Chapter 8 Propositions 228
Exercises 230 References 231

9 *Linear Systems of Simultaneous Equations* 233

- 9.1. Simultaneous Equations Bias 233
- 9.2. Instrumental Variables and Two-Stage Least Squares 238

- 9.3. Identification 243
- 9.4. Full-Information Maximum Likelihood Estimation 247
- 9.5. Estimation Based on the Reduced Form 250
- 9.6. Overview of Simultaneous Equations Bias 252

APPENDIX 9.A. Proofs of Chapter 9 Proposition 253
Exercise 255 References 256

10 *Covariance-Stationary Vector Processes* 257

- 10.1. Introduction to Vector Autoregressions 257
- 10.2. Autocovariances and Convergence Results for Vector Processes 261
- 10.3. The Autocovariance-Generating Function for Vector Processes 266
- 10.4. The Spectrum for Vector Processes 268
- 10.5. The Sample Mean of a Vector Process 279

APPENDIX 10.A. Proofs of Chapter 10 Propositions 285
Exercises 290 References 290

11 *Vector Autoregressions* 291

- 11.1. Maximum Likelihood Estimation and Hypothesis Testing for an Unrestricted Vector Autoregression 291
- 11.2. Bivariate Granger Causality Tests 302
- 11.3. Maximum Likelihood Estimation of Restricted Vector Autoregressions 309
- 11.4. The Impulse-Response Function 318
- 11.5. Variance Decomposition 323
- 11.6. Vector Autoregressions and Structural Econometric Models 324
- 11.7. Standard Errors for Impulse-Response Functions 336

APPENDIX 11.A. Proofs of Chapter 11 Propositions 340
APPENDIX 11.B. Calculation of Analytic Derivatives 344
Exercises 348 References 349

12 *Bayesian Analysis* 351

- 12.1. Introduction to Bayesian Analysis 351
- 12.2. Bayesian Analysis of Vector Autoregressions 360
- 12.3. Numerical Bayesian Methods 362

APPENDIX 12.A. *Proofs of Chapter 12 Propositions* 366
Exercise 370 *References* 370

13 *The Kalman Filter* 372

- 13.1. The State-Space Representation of a Dynamic System 372
- 13.2. Derivation of the Kalman Filter 377
- 13.3. Forecasts Based on the State-Space Representation 381
- 13.4. Maximum Likelihood Estimation of Parameters 385
- 13.5. The Steady-State Kalman Filter 389
- 13.6. Smoothing 394
- 13.7. Statistical Inference with the Kalman Filter 397
- 13.8. Time-Varying Parameters 399

APPENDIX 13.A. *Proofs of Chapter 13 Propositions* 403
Exercises 406 *References* 407

14 *Generalized Method of Moments* 409

- 14.1. Estimation by the Generalized Method of Moments 409
- 14.2. Examples 415
- 14.3. Extensions 424
- 14.4. *GMM* and Maximum Likelihood Estimation 427

APPENDIX 14.A. *Proofs of Chapter 14 Propositions* 431
Exercise 432 *References* 433

15 *Models of Nonstationary Time Series* 435

- 15.1. Introduction 435
- 15.2. Why Linear Time Trends and Unit Roots? 438

- 15.3. Comparison of Trend-Stationary and Unit Root Processes 438
- 15.4. The Meaning of Tests for Unit Roots 444
- 15.5. Other Approaches to Trended Time Series 447

**APPENDIX 15.A. Derivation of Selected Equations
for Chapter 15 451**
References 452

16 Processes with Deterministic Time Trends 454

- 16.1. Asymptotic Distribution of *OLS* Estimates of the Simple Time Trend Model 454
- 16.2. Hypothesis Testing for the Simple Time Trend Model 461
- 16.3. Asymptotic Inference for an Autoregressive Process Around a Deterministic Time Trend 463

**APPENDIX 16.A. Derivation of Selected Equations
for Chapter 16 472**
Exercises 474 References 474

17 Univariate Processes with Unit Roots 475

- 17.1. Introduction 475
- 17.2. Brownian Motion 477
- 17.3. The Functional Central Limit Theorem 479
- 17.4. Asymptotic Properties of a First-Order Autoregression when the True Coefficient Is Unity 486
- 17.5. Asymptotic Results for Unit Root Processes with General Serial Correlation 504
- 17.6. Phillips-Perron Tests for Unit Roots 506
- 17.7. Asymptotic Properties of a *p*th-Order Autoregression and the Augmented Dickey-Fuller Tests for Unit Roots 516
- 17.8. Other Approaches to Testing for Unit Roots 531
- 17.9. Bayesian Analysis and Unit Roots 532

APPENDIX 17.A. Proofs of Chapter 17 Propositions 534
Exercises 537 References 541

18 *Unit Roots in Multivariate Time Series* 544

- 18.1. Asymptotic Results for Nonstationary Vector Processes 544
- 18.2. Vector Autoregressions Containing Unit Roots 549
- 18.3. Spurious Regressions 557

APPENDIX 18.A. *Proofs of Chapter 18 Propositions* 562
Exercises 568 *References* 569

19 *Cointegration* 571

- 19.1. Introduction 571
- 19.2. Testing the Null Hypothesis of No Cointegration 582
- 19.3. Testing Hypotheses About the Cointegrating Vector 601

APPENDIX 19.A. *Proofs of Chapter 19 Propositions* 618
Exercises 625 *References* 627

20 *Full-Information Maximum Likelihood Analysis of Cointegrated Systems* 630

- 20.1. Canonical Correlation 630
- 20.2. Maximum Likelihood Estimation 635
- 20.3. Hypothesis Testing 645
- 20.4. Overview of Unit Roots—To Difference or Not to Difference? 651

APPENDIX 20.A. *Proofs of Chapter 20 Propositions* 653
Exercises 655 *References* 655

21 *Time Series Models of Heteroskedasticity* 657

- 21.1. Autoregressive Conditional Heteroskedasticity (ARCH) 657
- 21.2. Extensions 665

APPENDIX 21.A. *Derivation of Selected Equations for Chapter 21* 673
References 674

Difference Equations

1.1. First-Order Difference Equations

This book is concerned with the dynamic consequences of events over time. Let's say we are studying a variable whose value at date t is denoted y_t . Suppose we are given a dynamic equation relating the value y takes on at date t to another variable w_t and to the value y took on in the previous period:

$$y_t = \phi y_{t-1} + w_t. \quad [1.1.1]$$

Equation [1.1.1] is a *linear first-order difference equation*. A *difference equation* is an expression relating a variable y_t to its previous values. This is a *first-order* difference equation because only the first lag of the variable (y_{t-1}) appears in the equation. Note that it expresses y_t as a linear function of y_{t-1} and w_t .

An example of [1.1.1] is Goldfeld's (1973) estimated money demand function for the United States. Goldfeld's model related the log of the real money holdings of the public (m_t) to the log of aggregate real income (I_t), the log of the interest rate on bank accounts (r_{bt}), and the log of the interest rate on commercial paper (r_{ct}):

$$m_t = 0.27 + 0.72m_{t-1} + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}. \quad [1.1.2]$$

This is a special case of [1.1.1] with $y_t = m_t$, $\phi = 0.72$, and

$$w_t = 0.27 + 0.19I_t - 0.045r_{bt} - 0.019r_{ct}.$$

For purposes of analyzing the dynamics of such a system, it simplifies the algebra a little to summarize the effects of all the input variables (I_t , r_{bt} , and r_{ct}) in terms of a scalar w_t as here.

In Chapter 3 the input variable w_t will be regarded as a random variable, and the implications of [1.1.1] for the statistical properties of the output series y_t will be explored. In preparation for this discussion, it is necessary first to understand the mechanics of difference equations. For the discussion in Chapters 1 and 2, the values for the input variable $\{w_1, w_2, \dots\}$ will simply be regarded as a sequence of deterministic numbers. Our goal is to answer the following question: If a dynamic system is described by [1.1.1], what are the effects on y of changes in the value of w ?

Solving a Difference Equation by Recursive Substitution

The presumption is that the dynamic equation [1.1.1] governs the behavior of y for all dates t . Thus, for each date we have an equation relating the value of

y for that date to its previous value and the current value of w :

<i>Date</i>	<i>Equation</i>	
0	$y_0 = \phi y_{-1} + w_0$	[1.1.3]
1	$y_1 = \phi y_0 + w_1$	[1.1.4]
2	$y_2 = \phi y_1 + w_2$	[1.1.5]
\vdots	\vdots	
t	$y_t = \phi y_{t-1} + w_t$	[1.1.6]

If we know the starting value of y for date $t = -1$ and the value of w for dates $t = 0, 1, 2, \dots$, then it is possible to simulate this dynamic system to find the value of y for any date. For example, if we know the value of y for $t = -1$ and the value of w for $t = 0$, we can calculate the value of y for $t = 0$ directly from [1.1.3]. Given this value of y_0 and the value of w for $t = 1$, we can calculate the value of y for $t = 1$ from [1.1.4]:

$$y_1 = \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1,$$

or

$$y_1 = \phi^2 y_{-1} + \phi w_0 + w_1.$$

Given this value of y_1 and the value of w for $t = 2$, we can calculate the value of y for $t = 2$ from [1.1.5]:

$$y_2 = \phi y_1 + w_2 = \phi(\phi^2 y_{-1} + \phi w_0 + w_1) + w_2,$$

or

$$y_2 = \phi^3 y_{-1} + \phi^2 w_0 + \phi w_1 + w_2.$$

Continuing recursively in this fashion, the value that y takes on at date t can be described as a function of its initial value y_{-1} and the history of w between date 0 and date t :

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \dots + \phi w_{t-1} + w_t. \quad [1.1.7]$$

This procedure is known as solving the difference equation [1.1.1] by *recursive substitution*.

Dynamic Multipliers

Note that [1.1.7] expresses y_t as a linear function of the initial value y_{-1} and the historical values of w . This makes it very easy to calculate the effect of w_0 on y_t . If w_0 were to change with y_{-1} and w_1, w_2, \dots, w_t taken as unaffected, the effect on y_t would be given by

$$\frac{\partial y_t}{\partial w_0} = \phi^t. \quad [1.1.8]$$

Note that the calculations would be exactly the same if the dynamic simulation were started at date t (taking y_{t-1} as given); then y_{t+j} could be described as a

function of y_{t-1} and $w_t, w_{t+1}, \dots, w_{t+j}$:

$$y_{t+j} = \phi^{j+1}y_{t-1} + \phi^j w_t + \phi^{j-1}w_{t+1} + \phi^{j-2}w_{t+2} + \dots + \phi w_{t+j-1} + w_{t+j}. \quad [1.1.9]$$

The effect of w_t on y_{t+j} is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j. \quad [1.1.10]$$

Thus the *dynamic multiplier* [1.1.10] depends only on j , the length of time separating the disturbance to the input (w_t) and the observed value of the output (y_{t+j}). The multiplier does not depend on t ; that is, it does not depend on the dates of the observations themselves. This is true of any linear difference equation.

As an example of calculating a dynamic multiplier, consider again Goldfeld's money demand specification [1.1.2]. Suppose we want to know what will happen to money demand two quarters from now if current income I_t were to increase by one unit today with future income I_{t+1} and I_{t+2} unaffected:

$$\frac{\partial m_{t+2}}{\partial I_t} = \frac{\partial m_{t+2}}{\partial w_t} \times \frac{\partial w_t}{\partial I_t} = \phi^2 \times \frac{\partial w_t}{\partial I_t}.$$

From [1.1.2], a one-unit increase in I_t will increase w_t by 0.19 units, meaning that $\partial w_t / \partial I_t = 0.19$. Since $\phi = 0.72$, we calculate

$$\frac{\partial m_{t+2}}{\partial I_t} = (0.72)^2(0.19) = 0.098.$$

Because I_t is the log of income, an increase in I_t of 0.01 units corresponds to a 1% increase in income. An increase in m_t of $(0.01) \cdot (0.098) \cong 0.001$ corresponds to a 0.1% increase in money holdings. Thus the public would be expected to increase its money holdings by a little less than 0.1% two quarters following a 1% increase in income.

Different values of ϕ in [1.1.1] can produce a variety of dynamic responses of y to w . If $0 < \phi < 1$, the multiplier $\partial y_{t+j} / \partial w_t$ in [1.1.10] decays geometrically toward zero. Panel (a) of Figure 1.1 plots ϕ^j as a function of j for $\phi = 0.8$. If $-1 < \phi < 0$, the multiplier $\partial y_{t+j} / \partial w_t$ will alternate in sign as in panel (b). In this case an increase in w_t will cause y_t to be higher, y_{t+1} to be lower, y_{t+2} to be higher, and so on. Again the absolute value of the effect decays geometrically toward zero. If $\phi > 1$, the dynamic multiplier increases exponentially over time as in panel (c). A given increase in w_t has a larger effect the farther into the future one goes. For $\phi < -1$, the system [1.1.1] exhibits explosive oscillation as in panel (d).

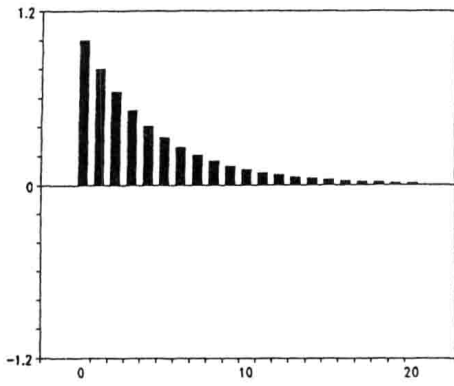
Thus, if $|\phi| < 1$, the system is stable; the consequences of a given change in w_t will eventually die out. If $|\phi| > 1$, the system is explosive. An interesting possibility is the borderline case, $\phi = 1$. In this case, the solution [1.1.9] becomes

$$y_{t+j} = y_{t-1} + w_t + w_{t+1} + w_{t+2} + \dots + w_{t+j-1} + w_{t+j}. \quad [1.1.11]$$

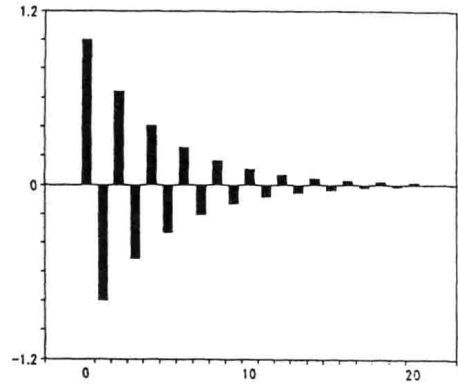
Here the output variable y is the sum of the historical inputs w . A one-unit increase in w will cause a permanent one-unit increase in y :

$$\frac{\partial y_{t+j}}{\partial w_t} = 1 \quad \text{for } j = 0, 1, \dots$$

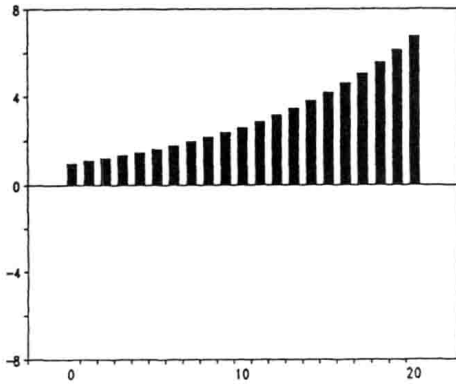
We might also be interested in the effect of w on the present value of the stream of future realizations of y . For a given stream of future values $y_t, y_{t+1},$



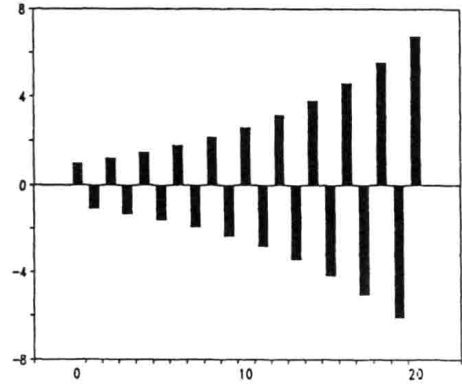
(a) $\phi = 0.8$



(b) $\phi = -0.8$



(c) $\phi = 1.1$



(d) $\phi = -1.1$

FIGURE 1.1 Dynamic multiplier for first-order difference equation for different values of ϕ (plot of $\partial y_{t+j}/\partial w_t = \phi^j$ as a function of the lag j).

y_{t+2}, \dots and a constant interest rate¹ $r > 0$, the *present value* of the stream at time t is given by

$$y_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \frac{y_{t+3}}{(1+r)^3} + \dots \quad [1.1.12]$$

Let β denote the discount factor:

$$\beta \equiv 1/(1+r).$$

Note that $0 < \beta < 1$. Then the present value [1.1.12] can be written as

$$\sum_{j=0}^{\infty} \beta^j y_{t+j}. \quad [1.1.13]$$

Consider what would happen if there were a one-unit increase in w_t , with w_{t+1}, w_{t+2}, \dots unaffected. The consequences of this change for the present value of y are found by differentiating [1.1.13] with respect to w_t and then using [1.1.10]

¹The interest rate is measured here as a fraction of 1; thus $r = 0.1$ corresponds to a 10% interest rate.