SMM 6

Surveys of Modern Mathematics

Open Problems and Surveys of Contemporary Mathematics

微分几何未解决问题及当代数学概观

Editors: Lizhen Ji · Yat-Sun Poon · Shing-Tung Yau

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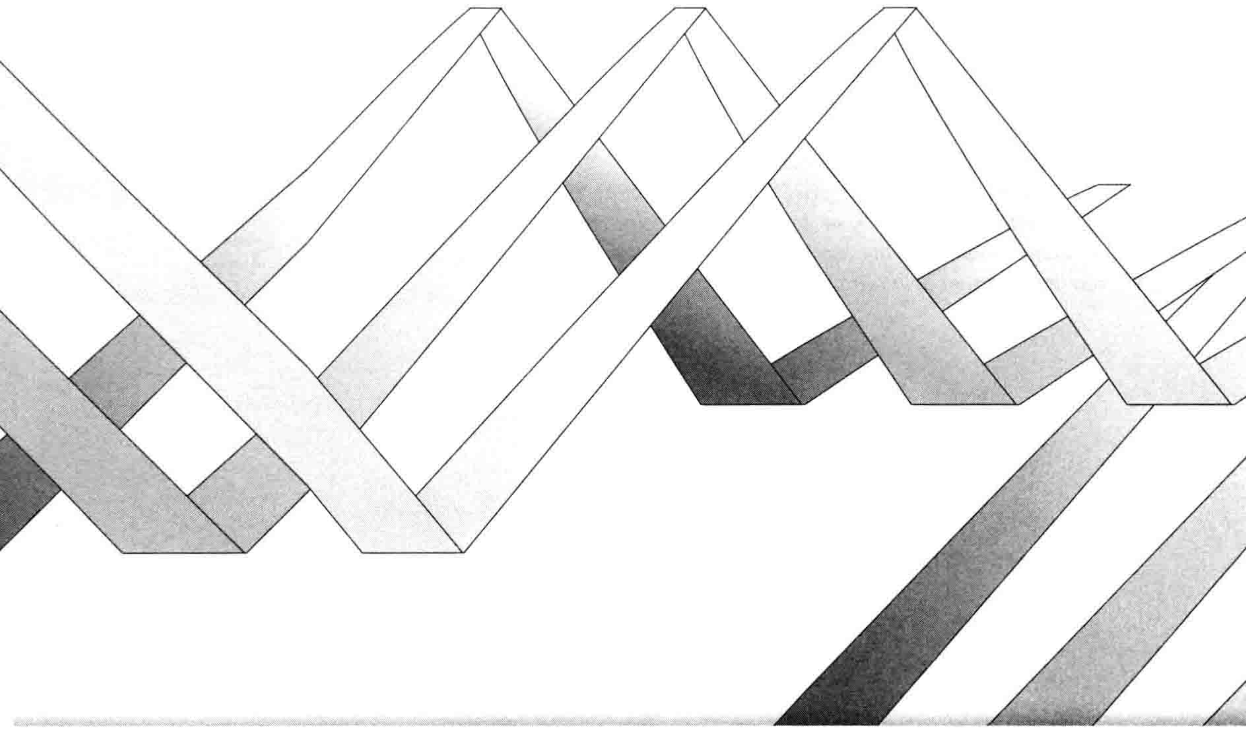

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WEIFEN JIHE WEI JIEJUE WENTI JI DANGDAI SHUXUE GAIGUAN

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Mathematics has developed to a very high level and is still developing rapidly. An important feature of the modern mathematics is strong interaction between different areas of mathematics. It is both fruitful and beautiful. For further development in mathematics, it is crucial to educate students and younger generations of mathematicians about important theories and recent developments in mathematics. For this purpose, accessible books that instruct and inform the reader are crucial. This new book series "Surveys of Modern Mathematics" (SMM) is especially created with this purpose in mind. Books in SMM will consist of either lecture notes of introductory courses, collections of survey papers, expository monographs on well-known or developing topics.

With joint publication by Higher Education Press (HEP) inside China and International Press (IP) in the West with affordable prices, it is expected that books in this series will broadly reach out to the reader, in particular students, around the world, and hence contribute to mathematics and the world mathematics community.

Series Editors

Preface

This book consists mainly of lecture notes of some talks and courses at Mathematical Sciences Center (MSC) of Tsinghua University together with several other papers.

Since it was founded in December 2009, one of the missions of MSC has been to teach both undergraduate and graduate students important ideas, theories and results of contemporary mathematics. These lecture notes reflect this philosophy. They are expository and accessible to both students and nonexperts. On the other hand, they also contain novel ideas or presentations of important topics in mathematics. Therefore, this book is also useful to experts. Especially we would like to point out that the last paper in this book *Open Problems in Differential Geometry* by the third editor of this book is only the first three of many lectures given by him in both Beijing and Taipei, which can be considered as a reviewing and updating of the very influential open problem lists by him.

Besides these lecture notes from MSC, this book also contains four other papers. The first is a paper by James Milne based on his talk at the seminar "What is ..." at University of Michigan. The concept of motives is important and difficult, and the talk and this paper are attempts by an expert to explain it in concrete terms. The second is a master thesis in 2002 by Joris van Hoboken who gives a coherent and accessible exposition of the ubiquity of the important ADE classification in mathematics, which originally occurred in the classification of simple complex Lie algebras. Joris van Hoboken switched to study law right after obtaining his Master degree and is now a senior researcher at a law school. The ADE classification occurs at many different situations, and it is still a mystery whether there are some deep, intrinsic connections between them. This master thesis was never published and has been highly cited and circulated on the web. We are grateful that Dr. van Hoboken has given us permission to include it in the current book. We hope that this will make the ADE classification better known to the reader and also give a permanent record of this beautiful master thesis. The other two are reprints of papers of the third editor. The short paper *A note on the distribution of critical points of eigenfunctions* considered a novel question. As it is well-known, the location and distribution of the zero sets (i.e., nodal sets) of eigenfunctions of Riemannian manifolds have been extensively and intensively studied. Critical points of eigenfunctions are also special and deserve to be understood better. Analysis on nonsmooth spaces has been becoming quite important and applied to several subjects in mathematics. The paper is one of the early papers in this subject.[1] Due to inaccessibility and no review of it in MathSciNet, this paper has been largely unknown. We hope that its inclusion in this book will be valuable to the reader as well.

It has been a lot of work for the speakers at MSC to write up their lecture notes. We would like to thank them, especially the four note-takers and co-authors (Hui Ma, Chun-Jun Tsai, Mu-Tao Wang, En-Tao Zhao) of the last paper in this

[1] For a recent survey, see J. Heinonen, *Nonsmooth calculus*. Bull. Amer. Math. Soc. (N.S.) 44 (2007), no. 2, 163–232.

book, for their efforts and contributions. We would also like to thank reviewers of the papers in this book for their help.

This book marks the beginning of publication from MSC and we hope and expect that future volumes will appear regularly.

Editors: Lizhen Ji, Yat-Sun Poon, Shing-Tung Yau

May 30, 2013

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Lectures on Mirror Symmetry and Topological String Theory

Murad Alim*

Abstract

These are notes of a series of lectures on mirror symmetry and topological string theory given at the Mathematical Sciences Center of Tsinghua University. The $\mathcal{N} = 2$ superconformal algebra, its deformations and its chiral ring are reviewed. A topological field theory can be constructed whose observables are only the elements of the chiral ring. When coupled to gravity, this leads to topological string theory. The identification of the topological string A- and B-models by mirror symmetry leads to surprising connections in mathematics and provides tools for exact computations as well as new insights in physics. A recursive construction of the higher genus amplitudes of topological string theory expressed as polynomials is reviewed.

2000 Mathematics Subject Classification: 14J33, 81T30, 81T45

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Contents

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1 Introduction

The study of supersymmetric gauge theories and string theory has been a very fruitful arena of interaction between mathematics and physics which has pushed the boundaries of knowledge and understanding on both sides. A key lesson learned on the physics side is that it is very useful to think of physical theories in terms of families, where members of a family of theories are related by deformations. This amounts to studying theories together with their space of couplings. Physical theories are usually formulated using a Lagrangian for the light degrees of freedom and their interactions. When the interactions are governed by a small coupling, perturbative techniques can be used to evaluate the correlation functions of the physical observables. Perturbation theory does however not take into account non-perturbative effects which are nontrivial field configurations with finite action which have to be considered as various backgrounds for perturbation theory and are in general hard to study. As the coupling of the theory is varied, perturbation theory breaks down as the non-perturbative effects become more and more important signaling that the original choice of light degrees of freedom is no longer valid. Hence, in different regions in the coupling space, perturbative and non-perturbative degrees of freedom may interchange.

A great leap forward in the understanding of the interplay of perturbative and non-perturbative degrees of freedom and the physical manifestation of their interchange in terms of dualities followed the pioneering work of Seiberg and Witten (SW) [1]. In this work the exact low energy effective theory was determined including all non-perturbative effects using holomorphicity and the expected behavior at singularities in the space of the effective couplings due to physical particles becoming massless. The key insight was that the problem of determining the exact couplings could be mapped to an equivalent mathematical problem of determining periods of a curve. This auxiliary geometry could be given a physical meaning, identifying it with part of the Calabi-Yau (CY) threefold geometry of a type IIB string compactification [2], embedding this understanding into the larger framework of string dualities (See [3] for a review).

One particular physical duality which had emerged earlier is mirror symmetry. It was observed that an exchange of signs of some generators of the $\mathcal{N} = (2,2)$ superconformal algebra underlying some string compactifications leads to an isomorphism [4]. This isomorphism identifies furthermore a ring of operators in the superconformal algebra, the chiral ring [5]. The deformation problem and the isomorphism defined abstractly using the superconformal algebra can be given a mathematical meaning when this algebra is realized by nonlinear sigma models defined on different target spaces Z and Z^*, see [6] and references therein.[1] Truncating the states of a representation of the superconformal algebra to those created by operators in the chiral ring and summing over all $2d$ topologies defines topological string theory, the two isomorphic versions are called the A- or the B-model respectively. These probe Kähler and complex structure deformations of mirror Calabi-Yau threefolds Z and Z^*. The B-model is more accessible to computations since its deformations are the complex structure deformations of Z^* which are captured by the variation of Hodge structure. Mirror symmetry is established by providing the mirror maps which are a distinguished set of local coordinates in a given patch of the deformation space. These provide the map to the A-model, since they are naturally associated with the chiral ring.

At special loci in the moduli space, the A-model data provides enumerative information of the CY Z. Using methods [11] to construct the mirror manifold of the quintic, it was thus possible [12] to compute the data associated to the variation of Hodge structure in the B-model and make predictions of the Gromov-Witten (GW) invariants. The higher genus GW invariants can be resumed to give integer multiplicities of BPS states in a five-dimensional theory obtained from an M-theory compactification on Z [13, 14]. Moreover, the special geometry governing the deformation spaces allows one to compute the prepotential $F_0(t)$ which governs the exact effective action of the four dimensional theories obtained from compactifying type IIA(IIB) string theory on $Z(Z^*)$, respectively. Various $4d$ gauge theories can be geometrically engineered [15, 16] and mirror symmetry can be used to compute their exact effective actions. (See also [17] and references therein.)

The prepotential is the genus zero free energy of topological string theory,

[1] By now there is a vast literature on the subject of mirror symmetry, see in particular [7, 8, 9, 10] and references therein for a more complete account of the developments leading to mirror symmetry.

which is defined perturbatively in a coupling constant governing the higher genus expansion. The partition function of topological string theory indicating its dependence on local coordinates in the deformation space has the form:

$$\mathcal{Z}(t,\bar{t}) = \exp\left(\sum_g \lambda^{2g-2}\mathcal{F}^{(g)}(t,\bar{t})\right) . \tag{1.1}$$

In [18, 19], Bershadsky, Cecotti, Ooguri and Vafa (BCOV) developed the theory and properties of the higher genus topological string free energies putting forward recursive equations, the holomorphic anomaly equations along with a method to solve these in terms of Feynman diagrams. For the full partition function these equations take the form of a heat equation [19, 20] and can be interpreted [20] as describing the background independence of the partition function when the latter is interpreted as a wave function associated to the geometric quantization of $H^3(Z^*)$.

Using the special geometry of the deformation space a polynomial structure of the higher genus amplitudes in a finite number of generators was proven for the quintic and related one parameter deformation families [21] and generalized to arbitrary target CY manifolds [22], see also [23, 24, 25]. The polynomial structure supplemented by appropriate boundary conditions enhances the computability of higher genus amplitudes. Moreover the polynomial generators are expected to bridge the gap towards constructing the appropriate modular forms for a given target space duality group which is reflected by the special geometry of the CY manifold.

The aim of these notes is to provide an accessible concise introduction into some of the ideas outlined above. The presentation is far from being self-contained and the topics which are covered represent a small sample of a vast amount of developments in this field, reference will be given to the original literature and to many of the excellent reviews on the subject. By the nature of it, the material presented here is very close to own contributions and in fact it relies heavily on [22, 26, 27, 28].

The outline of these notes is as follows. In Sect. 2 the $\mathcal{N} = 2$ superconformal algebra and its chiral ring as well as its deformations are reviewed. It is discussed how to construct a bundle of states out of the subring of the chiral ring spanned by the deformation operators. Furthermore, the geometric realization of the $\mathcal{N} = (2,2)$ superconformal algebra as a nonlinear sigma model is described. In Sect. 3 it is outlined how topological string theory can be obtained by first restricting the physical observables to the chiral ring and then coupling to gravity. The two inequivalent ways of doing so, the A- and B-model are discussed as well as the geometric interpretation of the chiral ring on both sides. Special geometry and its B-model realization in the variation of Hodge structure are reviewed. The quintic and its mirror are discussed as an example of constructing mirror geometries and finding the mirror map. In Sect. 4, the BCOV anomaly equation and its recursive solution are reviewed. Using special geometry, a polynomial structure of the higher genus amplitudes can be proven and used for easier computations when the physically expected boundary conditions are implemented. An example of applying this structure to the quintic is given.

2 The superconformal algebra and the chiral ring

Mirror symmetry originates from representations of the $\mathcal{N} = (2,2)$ superconformal algebra, which refers to two copies, left- and right-moving of the $\mathcal{N} = 2$ superconformal algebra which is discussed in the following. The topological string A- and B-models which are identified by mirror symmetry refer to a truncation of the states of a representation of $\mathcal{N} = (2,2)$ superconformal algebra to the chiral ring [5]. The following is based on [27, 29, 6, 30, 31, 10, 32].

2.1 $\mathcal{N} = 2$ superconformal algebra

The $\mathcal{N} = 2$ superconformal algebra is generated by the energy momentum tensor $T(z)$, which has conformal weight $h = 2$, by two anti-commuting currents $G^{\pm}(z)$ of conformal weight $3/2$ and a $U(1)$ current $J(z)$ under which the $G^{\pm}(z)$ carry charge $\pm$ and z is a local coordinate on the $2d$ world-sheet. These currents satisfy the following operator product expansions (OPE):

$$G^{\pm}(z)G^{\mp}(w) = \frac{\frac{2}{3}c}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm \partial_w J(w)}{(z-w)} + \cdots , \tag{2.1}$$

$$J(z)\, G^{\pm}(w) = \pm \frac{G^{\pm}(w)}{(z-w)} + \cdots , \tag{2.2}$$

$$J(z)\, J(w) = \frac{\frac{1}{3}c}{(z-w)^2} + \cdots , \tag{2.3}$$

$$T(z)\, J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \cdots , \tag{2.4}$$

$$T(z)\, G^{\pm}(w) = \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial_w G^{\pm}(w)}{(z-w)} + \cdots , \tag{2.5}$$

$$T(z)\, T(w) = \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots , \tag{2.6}$$

where the dots refer to the addition of regular terms in the limit $z \to w$, c is the central charge. The boundary conditions which must be imposed for the currents $G^{\pm}(z)$ can be summarized as follows

$$G^{\pm}(e^{2\pi i}z) = -e^{\mp 2\pi i a} G^{\pm}(z) , \tag{2.7}$$

with a continuous parameter a which lies in the range $0 \leq a < 1$. For integral and half integral a one recovers anti-periodic and periodic boundary conditions corresponding to the Ramond and Neveu-Schwarz sectors, respectively. The currents can be expanded in Fourier modes

$$T(z)=\sum_n \frac{L_n}{z^{n+2}}\,,\quad G^{\pm}(z)=\sum_n \frac{G^{\pm}_{n\pm a}}{z^{n\pm a+\frac{3}{2}}}\,,\quad J(z)=\sum_n \frac{J_n}{z^{n+1}}\,. \tag{2.8}$$

The $\mathcal{N}=2$ superconformal algebra can be expressed in terms of the operator product expansion of the currents or by the commutation relations of their modes. The latter reads:

$$\begin{aligned}
[L_m,L_n] &= (m-n)L_{m+n}+\frac{c}{12}m(m^2-1)\delta_{m+n,0}\,,\\
[J_m,J_n] &= \frac{c}{3}m\delta_{m+n,0}\,,\\
[L_n,J_m] &= -mJ_{m+n}\,,\\
\left[L_n,G^{\pm}_{m\pm a}\right] &= \left(\frac{n}{2}-(m\pm a)\right)G^{\pm}_{m+n\pm a}\,,\\
\left[J_n,G^{\pm}_{m\pm a}\right] &= \pm G^{\pm}_{n+m\pm a}\,,\\
\left\{G^{+}_{n+a},G^{-}_{m-a}\right\} &= 2L_{m+n}+(n-m+2a)J_{n+m}\\
&\quad +\frac{c}{3}\left((n+a)^2-\frac{1}{4}\right)\delta_{m+n,0}\,.
\end{aligned} \tag{2.9}$$

The algebras obtained for every value of the continuous parameter a are isomorphic. This isomorphism induces an operation on the states which is called *spectral flow*. This operation shows the equivalence of the Ramond (R) and Neveu-Schwarz (NS) sectors as the states in each sector are continuously related by the flow. Moreover, as NS and R sectors give rise to space-time bosons and space-time fermions respectively, this isomorphism of the algebra induces space-time supersymmetry.

2.2 Chiral ring

The representation theory of the $\mathcal{N}=2$ superconformal algebra is equipped with an interesting additional structure, namely a finite sub-sector of the operators creating the highest weight states carries an additional ring structure which is discussed in the following.

The unitary irreducible representations of the algebra can be built up from highest weight states by acting on these with creation operators which are identified with the negative modes in (2.8). Similarly all the modes with positive indices can be thought of as annihilation operators which lower the L_0 eigenvalue of a state. A highest weight state is thus one which satisfies,

$$L_n|\phi\rangle=0\,,\quad G^{\pm}_r|\phi\rangle=0\,,\quad J_m|\phi\rangle=0\,,\quad n,r,m>0\,. \tag{2.10}$$

The zero index modes L_0 and J_0 modes can be used to label the states by their eigenvalues

$$L_0|\phi\rangle=h_\phi|\phi\rangle\,,\quad J_0|\phi\rangle=q_\phi|\phi\rangle\,. \tag{2.11}$$

In the Ramond sector there are furthermore the modes $G_0^{\pm}$. If a state is annihilated by these then it is called a Ramond ground state. A highest weight state is created by a primary field ϕ

$$\phi|0\rangle = |\phi\rangle\,. \tag{2.12}$$

The subset of primary fields which will be of interest is constituted of the *chiral primary* fields. States which are created by those satisfy furthermore

$$G^{+}_{-1/2}|\phi\rangle = 0\,. \tag{2.13}$$

The name anti-chiral primary will be used for the primary fields annihilated by $G^{-}_{-1/2}$. In combination with the representations of the anti-holomorphic currents $\overline{G}^{\pm}$ this leads to the notions of (c,c), (a,c), (a,a), and (c,a) primary fields, where c and a stand for chiral and anti-chiral and the pair denotes the conditions in the holomorphic and anti-holomorphic sectors. Considering

$$\langle\phi|\{G^{-}_{1/2}, G^{+}_{-1/2}\}|\phi\rangle = ||G^{+}_{-1/2}|\phi\rangle||^2 = \langle\phi|2L_0 - J_0|\phi\rangle \geq 0\,, \tag{2.14}$$

implies

$$h_\phi \geq \frac{q_\phi}{2}\,, \tag{2.15}$$

with equality holding for chiral states. This property of chiral primary states is an analog of the BPS bound for physical states. Now looking at the operator product expansion of two chiral primary fields ϕ and χ

$$\phi(z)\chi(w) = \sum_i (z-w)^{h_{\psi_i}-h_\phi-h_\chi}\psi_i\,, \tag{2.16}$$

the $U(1)$ charges add $q_{\psi_i} = q_\phi + q_\chi$ and hence $h_{\psi_i} \geq h_\phi + h_\chi$. The operator product expansion has thus no singular terms and the only terms which survive in the expansion when $z \to w$ are the ones for which ψ_i is itself chiral primary. It is thus shown that the chiral primary fields give a closed non-singular ring under operator product expansion. Furthermore, one can show the finiteness of this ring by considering

$$\langle\phi|\{G^{-}_{3/2}, G^{+}_{-3/2}\}|\phi\rangle = \langle\phi|2L_0 - 3J_0 + \frac{2}{3}c|\phi\rangle \geq 0\,, \tag{2.17}$$

to see that the conformal weight of a chiral primary is bounded by $c/6$.

For the world-sheet description of certain string compactifications two copies of the $\mathcal{N}=2$ superconformal algebra are required, these are usually referred to as left and right moving or holomorphic and anti-holomorphic, the currents of the latter will be denoted by $\overline{T}, \overline{G}^{\pm}, \overline{J}$. The algebra is then referred to as the $\mathcal{N}=(2,2)$ superconformal algebra.

In the $\mathcal{N}=(2,2)$ superconformal algebra there are now four finite rings, depending on the different combinations of chiral and antichiral in the right and left moving algebras, this gives the (c,c), (a,c), (a,a) and (c,a) rings, one sees that the latter two are charge conjugates of the first two. The relation between charge and conformal weight for an anti-chiral primary becomes $h_\psi = -\frac{q_\psi}{2}$. Denoting the set of chiral primary fields by ϕ_i where the index i runs over all chiral primaries, the ring structure can be formulated as follows[2]

$$\phi_i \phi_j = C_{ij}^k \phi_k \,. \tag{2.18}$$

2.3 Deformation families

Mirror symmetry is a symmetry relating the deformation family of the (a,c) chiral ring with the deformation family of the (c,c) chiral ring. To make this statement more precise the deformation family of the (c,c) chiral ring will be discussed in the following. Deformations of a conformal field theory are achieved by adding marginal operators to the original action, these are operators having conformal weight $h + \overline{h} = 2$. In the following spinless operators will be studied which have $h = 1$, $\overline{h} = 1$. The operators which maintain their $h = \overline{h} = 1$ conformal weights after perturbation of the theory are called truly marginal operators. Such operators can be constructed from the chiral primary operators in two steps. For instance in the (c,c) ring, starting from an operator of charge $q = \overline{q} = 1$, $h = \overline{h} = 1/2$ one can first construct

$$\phi^{(1)}(w,\overline{w}) = \left[G^-(z), \phi(w,\overline{w})\right] = \oint dz\, G^-(z)\phi(w,\overline{w}) \,, \tag{2.19}$$

which now has $h = 1, q = 0$. In the next step

$$\phi^{(2)}(w,\overline{w}) = \left\{\overline{G}^-(z), \phi^{(1)}(w,\overline{w})\right\} = \oint d\overline{z}\, \overline{G}^-(\overline{z})\phi^{(1)}(w,\overline{w}) \,, \tag{2.20}$$

which has $h = \overline{h} = 1$ and zero charge and is hence a truly marginal operator and can be used to perturb the action of the theory

$$\delta S = t^i \int \phi_i^{(2)} + \overline{t}^{\overline{\imath}} \int \phi_{\overline{\imath}}^{(2)} \,, \quad i = 1, \ldots, n \,, \tag{2.21}$$

where a priori also the deformations coming from the (a,a) operators are included and $n = \dim \mathcal{H}^{(1,1)}$ denotes the dimension of the subspace of the Hilbert space of the theory spanned by the states which are created by the charge $(1,1)$ operators. A similar construction can be done for the (a,c) chiral ring. The superscript notation is borrowed from topological field theories where an analogous construction gives the two form descendants which can be used to perturb the topological theory, see for example [33]. The deformations constructed in this fashion span a deformation space $\mathcal{M}$, the moduli space of the SCFT.

[2] Formulas for products of operators are understood to hold within correlation functions.