J. Frank Adams

# Stable Homotopy Theory

Lecture Notes in Mathematics

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# Lecture Notes in Mathematics

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1) Introduction

Before I get down to the business of exposition, I'd like to offer a little motivation. I want to show that there are one or two places in homotopy theory where we strongly suspect that there is something systematic going on, but where we are not yet sure what the system is.

The first question concerns the stable J-homomorphism. I recall that this is a homomorphism

$$J: \pi_r(SO) \to \pi_r^S = \pi_{r+n}(S^n), \text{ n large.}$$

It is of interest to the differential topologists. Since Bott, we know that $\pi_r(SO)$ is periodic with period 8:

| $r =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9... |
|---|---|---|---|---|---|---|---|---|---|
| $\pi_r(SO) =$ | $Z_2$ | 0 | Z | 0 | 0 | 0 | Z | $Z_2$ | $Z_2$... |

On the other hand, $\pi_r^S$ is not known, but we can nevertheless ask about the behavior of J. The differential topologists prove:

Theorem: If $r = 4k - 1$, so that $\pi_r(SO) \cong Z$, then $J(\pi_r(SO)) = Z_m$ where m is a multiple of the denominator of $B_k/4k$ ($B_k$ being in the $k^{th}$ Bernoulli number.)

Conjecture: The above result is best possible, i.e. $J(\pi_r(SO)) = Z_m$ where m is exactly this denominator.

Status of conjecture: No proof in sight.

Conjecture: If $r = 8k$ or $8k + 1$, so that $\pi_r(SO) = Z_2$, then $J(\pi_r(SO)) = Z_2$.

Status of conjecture: Probably provable, but this is work in progress.

The second question is somewhat related to the first; it concerns vector fields on spheres. We know that $S^n$ admits a continuous field of non-zero tangent vectors if and only if n is odd. We also know that if $n = 1,3,7$ then $S^n$ is parallelizable: that is, $S^n$ admits n continuous tangent vector fields which are linearly independent at every point. The question is then: for each n, what is the maximum number, r(n), such that $S^n$ admits r(n) continuous tangent vector fields that are linearly independent at every point? This is a very classical problem in the theory of fibre bundles. The best positive result is due to Hurwitz, Radon and Eckmann who construct a certain number of vector fields by algebraic methods. The number, $\rho(n)$, of fields which they construct is always one of the numbers for which $\pi_r(SO)$ is not zero (0,1,3,7,8,9,11....). To determine which, write $n + 1 = (2t + 1)2^{\nu}$: then $\rho(n)$ depends only on $\nu$ and increasing $\nu$ by one increases $\rho(n)$ to the next allowable value.

<u>Conjecture</u>: This result is best possible: i.e. $\rho(n) = r(n)$.

<u>Status of conjecture</u>: This has been confirmed by Toda for $\nu < 11$.

It seems best to consider separately the cases in which $\rho(n) = 8k - 1, 8k, 8k + 1, 8k + 3$. The most favourable case appears to be that in which $\rho(n) = 8k + 3$. I have a line of investigation which gives hope of proving that the result is best possible in this case.

Now, I. M. James has shown that if $S^{q-1}$ admits r-fields, then $S^{2q-1}$ admits r + 1 fields. Therefore the proposition that $\rho(n) = r(n)$ when $\rho(n) = 8k + 3$ would imply that $r(n) \leq \rho(n) + 1$ in the other three cases. This would seem to show that the result is in sight in these cases also: either one can try to refine the inference based on James' result or one can try to adapt the proof of the case $\rho(n) = 8k + 3$ to the case $\rho(n) = 8k + 1$.

2) Primary operations

It is good general philosophy that if you want to show that a geometrical construction is possible, you go ahead and perform it; but if you want to show that a proposed geometric construction is impossible, you have to find a topological invariant which shows the impossibility. Among topological invariants we meet first the homology and cohomology groups, with their additive and multiplicative structure. Afte that we meet cohomology operations, such as the celebrated Steenrod square. I recall that this is a homomorphism

$$Sq^i : H^n(X,Y;Z_2) \to H^{n+i}(X,Y;Z_2)$$

defined for each pair (X,Y) and for all non-negative integers i and n. ($H^n$ is to be interpreted as singular cohomology.) The Steenrod square enjoys the following properties:

1) Naturality: if $f:(\overline{X},\overline{Y}) \to (X,Y)$ is a map, then $f^*(Sq^i u) = Sq^i f^* u$.
2) Stability: if $\delta$: $H^n(Y;Z_2) \to H^{n+1}(X,Y;Z_2)$ is the coboundary homomorphism of the pair (X,Y), then $Sq^i(\delta u) = \delta(Sq^i u)$
3) Properties for small values of i.
   i) $Sq^0 u = u$
   ii) $Sq^1 u = \beta u$ where $\beta$ is the Bockstein coboundary associated with the exact sequence $0 \to Z_2 \to Z_4 \to Z_2 \to 0$.

4) Properties for small values of n.

i) if $n = i$ $Sq^i u = u^2$

ii) if $n < i$ $Sq^i u = 0$.

5) Cartan formula:

$$Sq^i(u\cdot v) = \sum_{j+k=i} (Sq^j u)\cdot(Sq^k v)$$

6) Adam relations: if $i < 2j$ then

$$Sq^i Sq^j = \sum_{\substack{k+\ell = i+j \\ k \geq 2\ell}} \lambda_{k,l}\, Sq^k Sq^\ell$$

where the $\lambda_{k,\ell}$ are certain binomial coefficients which one finds in Adam's paper [1].

References for these properties are found in Serre[2]. These properties are certainly sufficient to characterize the Steenrod squares axiomatically; as a matter of fact, it is sufficient to take fewer properties, namely 1, 2, and 4(i).

Perhaps one word about Steenrod's definition is in order. One begins by recalling that the cup-product of cohomology classes satisfies

7) $u\cdot v = (-1)^{pq} v\cdot u$ where

$u \in H^p(X;Z)$ and $v \in H^q(X;Z)$.

However the cup-product of cochains does not satisfy this rule. One way of proving this rule is to construct, more or less explicitly, a chain homotopy: to every pair of cochains, x, y, one assigns a cochain, usually written $x \smile_1 x$, so that

$$\delta(x \smile_1 y) = xy - (-1)^{pq} yx$$

if x and y are cocycles of dimension p and q respectively. Therefore if x is a mod 2 cocycle of dimension m $\delta(x \cup_1 x) = xx \pm xx = 0 \mod 2$. We define $Sq^{n-1}x = \{x \cup_1 x\}$, the mod 2 cohomology class of the cocycle $x \cup_1 x$. Steenrod's definition generalized this procedure.

The notion of a primary operation is a bit more general. Suppose given n,m,G,H where n,m are non-negative integers and G and H are abelian groups. Then a primary operation of type (n,m,G,H) would be a <u>function</u>

$$\phi: H^n(X,Y;G) \to H^m(X,Y;H)$$

defined for each pair (X,Y) and natural with respect to mappings of such pairs.

Similarly, we define a stable primary operation of degree i. This is a sequence of functions:

$$\phi_n: H^n(X,Y;G) \to H^{n+i}(X,Y;H)$$

defined for each n and each pair (X,Y) so that each function $\phi_n$ is natural and $\phi_{n+1}\delta = \delta\phi_n$ where $\delta$ is the coboundary homomorphism of the pair (X,Y). From what we have assumed it can be shown that each function $\phi_n$ is necessarily a homomorphism.

Now let's take $G = H = Z_2$. Then the stable primary operations form a set A, which is actually a graded algebra because two such operations can be added or composed in the obvious fashion. One should obviously ask, "What is the structure of A?"

Theorem 1. (Serre) A is generated by the Steenrod squares $Sq^i$.

(For this reason, A is usually called the Steenrod algebra, and the elements $a \in A$ are called Steenrod operations.)

More precisely, A has a $Z_2$-basis consisting of the operations

$$Sq^{i_1} Sq^{i_2} \ldots Sq^{i_t}$$

where $i_1, \ldots, i_t$ take all values such that

$$i_r \geq 2i_{r+1} \quad (1 \leq r < t) \quad \text{and} \quad i_t > 0.$$

The empty product is to be admitted and interpreted as the identity operation.

(The restriction $i_r \geq 2i_{r+1}$ is obviously sensible in view of property 6) listed above.) There is an analogous theorem in which $Z_2$ is replaced by $Z_p$.

Remark: The products $Sq^{i_1} \ldots Sq^{i_t}$ considered above are called admissible monomials. It is comparatively elementary to show that they are linearly independent operations. For example, take $X = \overset{n}{\underset{1}{X}} RP^\infty$, a Cartesian product of n copies of real (infinite dimensional) projective spaces: let $x_i \in H^1(RP^\infty; Z_2)$ be the generators in the separate factors ($i = 1, \ldots, n$), so that $H^*(X; Z_2)$ is a polynomial algebra generated by $x_1, \ldots, x_n$. Then Serre and Thom have shown that the admissible monomials of a given dimension d take linearly independent values on the class $x = x_1 \cdot x_2 \cdot \ldots \cdot x_n \in H^n(X;Z_2)$ if n is sufficiently large compared to d.

The computation of $Sq^{i_1} \ldots \cdot Sq^{i_t}$ on the class x is reduced by the Cartan formula to the computation of other iterated operations on the $x_i$'s themselves. Properties 3(i), 4(i) and (ii) imply that $Sq^0 x_i = x_i$, $Sq^1 x_i = x_i^2$, and $Sq^j x_i = 0$ for $j > 1$. The Cartan formula then allows us to compute iterated operations on the $x_i$'s. The details are omitted.

The substance, then, of Theorem 1 is that the admissible monomials span A. This is proved by using Eilenberg-MacLane spaces.

I recall that a space K is called an Eilenberg-MacLane space of type $(\pi,n)$--written $K \in K(\pi,n)$--if and only if

$$\pi_r(K) = \begin{cases} \pi & \text{if } r = n \\ 0 & \text{otherwise.} \end{cases}$$

It follows, by the Hurewicz Isomorphism Theorem (if $n > 1$) that $H_r(K) = 0$ for $r < n$ and $H_n(X) \approx \pi$. Hence $H^n(K;\pi) \approx$ Hom $(\pi,\pi)$, and $H^n(K;\pi)$ contains an element $b^n$, the fundamental class, corresponding to the identity homomorphism from $\pi$ to $\pi$.

Concerning such spaces K, we have

Lemma 1. Let (X,Y) be "good" pair (e.g. homotopy equivalent to a CW-pair.) Let Map $(X,Y;K,k_0)$ denote the set of homotopy classes of mappings from the pair (X,Y) to the pair $(K,k_0)$, $k_0$ being a point of K. Then this set is in one-to-one correspondence with $H^n(X,Y;\pi)$. The correspondence is given by assigning to each class, $\{f\}$, of maps the element $f^*b^n$.

This lemma is proved by obstruction theory and is classical, see e. g. [3].

<u>Lemma 2</u>. There is a one to one correspondence between cohomology operations $\phi$, as defined above, and elements $C^m$ of $H^m(G,n;H)$. The correspondence is given by $\phi \to \phi(b^n)$. The notation $H^m(G,n;H)$ means the cohomology groups (coefficients H) of an Eilenberg-MacLane space of type $(G,n)$, this depends only on G, n and H. $b^n$ is the fundamental class in $H^n(G,n;G)$.

This lemma follows from the first rather easily for "nice " pairs. But a general pair can be replaced by a C-W pair without affecting the singular cohomology.

There is a similar corollary for stable operations. In order to state it, I need to recall that if $K \in K(G,n)$ then its space of loops, $\Omega K$, is an Eilenberg-MacLane space of type $(G,n-1)$. The suspension $\sigma$: $H^m(K) \to (\Omega K)$ is defined as follows:

Let K denote the space of paths in K. Then we have $\pi$: $(LK,\Omega K) \to (K,pt)$, the map that assigns to each path its endpoint. The map $\sigma$ is the composition:

$$H^m(K) \leftarrow H^m(K,pt) \xrightarrow{\pi^*} H^m(LK,\Omega K) \xleftarrow{\delta} H^{m-1}(\Omega K).$$

The arrows which point the wrong way are conveniently isomorphisms so can be reversed, the last one, $\delta$, is such because LK is a contractible space.

Lemma 3. There is a 1-1 correspondence between stable primary operations, as considered above, and sequences of elements $e^{n+i} \in H^{n+i}(G,n;H)$ (one for each n) such that $\sigma e^{n+i} = e^{n-1+i}$.

We may rephrase this. For n sufficiently large the groups $H^{n+i}(G,n;H)$ may be identified under the map $\sigma$ for it is then an isomorphism. Any of these isomorphic groups can be called the "stable Eilenberg-MacLane group of degree i". The lemma then asserts that stable primary operations of degree i correspond one to one with the elements stable of the Eilenberg-MacLane group of degree i. For Theorem 1, then, it remains to calculate these groups in the case $G = H = Z_2$.

Theorem 2. (Serre) $H^*(Z_2,n;Z_2)$ is a polynomial algebra, having as generators the classes

$$Sq^{i_1}Sq^{i_2}\ldots Sq^{i_t}b^n$$

where $i_1,\ldots,i_t$ take all values such that

i) $i_1 \geq 2i_2,\ldots,i_{t-1} \geq 2i_t \quad i_t > 0$

ii) $i_r < i_{r+1} + \ldots + i_t + n$ for each r.

The empty sequence is again allowed and interpreted as indicating the fundamental class $b^n$.

Remark: These restrictions are obviously sensible in view of properties 4 and 6 above. The conditions are not all independent but this does not worry us.

The proof of the theorem proceeds by induction on n. We know that $H^*(K(Z_2,1);Z_2)$ is a polynomial algebra on one generator $b^1$ because $RP^\infty$ qualifies as a $K(Z_2,1)$. The inductive step consists in arguing from $H^*(Z_2,n;Z_2)$ to $H^*(Z_2,n+1;Z_2)$ by applying the little Borel theorem to the fibering $\Omega K \to LK \to K$ mentioned above where $K \in K(Z_2,n+1)$.

Let me recall the little Borel theorem.

Classes $f_1,f_2,\ldots,f_i,\ldots$ in $H^*(F;Z_2)$ are said to form a <u>simple system of generators</u> if and only if the products $f_1^{\epsilon_1} f_2^{\epsilon_2} \ldots f_n^{\epsilon_n}$ ($\epsilon_i = 0$, or 1) form a $Z_2$-basis for $H^*(F;Z_2)$.

<u>Theorem 3.</u> (Borel) Let $F \to E \to B$ be a fibration with B simply connected and E contractible. Let $b_1,b_2,\ldots$ be classes in $H^*(B;Z_2)$ such that only a finite number of them lie in any one group $H^n(B;Z_2)$ <u>and</u> such that $\{\sigma(b_i)\}$ is a simple system of generators in $H^*(F;Z_2)$. <u>Then</u> $H^*(B;Z_2)$ is a polynomial algebra generated by $b_1,b_2,\ldots$ .

For example, in $H^*(Z_2,1;Z_2)$ the classes $b^1$, $(b^1)^2$, $(b^1)^4$, $(b^1)^8$, ... form a simple system of generators. Also in $H^*(Z_2,2;Z_2)$ we have the classes $b^2$, $Sq^1b^2$, $Sq^2Sq^1b^2$, ... and

$$\sigma(b^2) = b^1$$

$$\sigma(Sq^1b^2 = Sq^1\sigma(b^2) = Sq^1(b^1) = (b^1)^2$$

$$\sigma(Sq^2Sq^1b^2) = Sq^2\sigma(Sq^1b^1) = Sq^2(b^1)^2 = (b^1)^4$$

etc.

Hence $H^*(Z_2,2;Z_2)$ is a polynomial algebra generated by $b^2$,

$b^2$, $Sq^1b^2$, $Sq^2Sq^1b^2$,... . In a similar way, one argues from $K(Z_2,n)$ to $K(Z_2,n+1)$.

The little Borel theorem is most conveniently proved by using the comparison theorem for spectral sequences. In fact, in the situation of the little Borel theorem, we have two spectral sequences: the first is the spectral sequence of the fibering, and the second is our idea of what the first ought to be. We wish to prove these coincide--which is just what the comparison theorem is for.

However, you have to choose your comparison theorem. The version given by John Moore [4] won't do, because in that version, you have to start on the chain level, and here we wish to start with the $E_2$ terms. The version given by Chris Zeeman [5] will do very nicely. Zeeman's proof, however, can be greatly simplified in the special case when the $E_\infty$ terms are trivial, and this is the case we need (in fact, it's the only case I've ever needed.)

Before stating the comparison theorem, we recall some notation. A spectral sequence contains a collection of groups $E_r^{p,q}$ $\infty \geq r \geq 2$, p,q integers (Ours will satisfy $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$.) It also contains differentials $d_r$: $E_r^{p,q} \to E_r^{p+r,\ q-r+1}$ such that $d_r \circ d_r = 0$ and such that $H(E_r^{**};d_r) = E_{r+1}^{**}$. A map, f, between one spectral sequence $\{E_r^{p,q}\}$ and another $\{\bar{E}_r^{p,q}\}$ is a collection of homomorphism $f\colon E_r^{p,q} \to \bar{E}_r^{p,q}$ which commute with the $d_r$'s in an obvious way.

Theorem 4. Comparison Theorem for Spectral Sequences. Let f be a map between two spectral sequences $E_r^{p,q}$ and $\overline{E}_r^{p,q}$ such that:

i) If $f\colon E_2^{p,o} \simeq \overline{E}_2^{p,o}$ for $p \leq P$

Then $f\colon E_2^{p,q} \simeq \overline{E}_2^{p,q}$ for $p \leq P$, all q

ii) $E_\infty^{p,q} = 0$ $\overline{E}_\infty^{p,q} = 0$ except for $(p,q) = 0,0)$ in which case

$$f\colon E_\infty^{0,0} \simeq \overline{E}_\infty^{0,0} .$$

Then $f\colon E_2^{p,o} \cong E_2^{p,o}$ for all p.

Proof: The proof is by induction on p. The result is true for both $p = 0$ and $p = 1$ by assumption because $E_\infty^{0,0} = E_2^{0,0}$ and $E_\infty^{1,0} = E_\infty^{1,0}$, and similarly for $\overline{E}$, and f is an isomorphism on these $E_\infty$ terms. Now assume that $f\colon E_2^{p,0} \simeq \overline{E}_2^{p,0}$ for $p \leq P$. Recall that $0 \subset B_2^{p,q} \subset Z_2^{p,q} \subset E_2^{p,q}$ where $B_2^{p,q} = \operatorname{Im} d_2$ and $Z_2^{p,q} = \operatorname{Ker} d_2$, and $H_2 = Z_2^{p,q}/B_2^{p,q} = E_3^{p,q}$ (The tedious superscripts will sometimes be omitted in what follows.) Since $d_3$ is defined on $E_3^{p,q}$, $\operatorname{Im} d_3$ and $\operatorname{Ker} d_3$ give rise to subgroups $B_3$ and $Z_3$ such that $0 \subset B_2 \subset B_3 \subset Z_3 \subset Z_2 \subset E_2^{p,q}$. This process continues; in general we have $0 = B_1 \subset B_2 \subset \cdots \subset B_p \subset Z_{q+1} \subset Z_q \cdots \subset Z_2 \subset Z_1 = E_2^{p,q}$. The quotient group $Z_{q+1}/B_p$ is $E_\infty^{p,q}$, hence zero in our case, at least if

$(p,q) \neq (0,0)$. The boundary map $d_r$ give an isomorphism

$$(Z_{r-1}/Z_r)^{p,q} \xrightarrow{\cong} (B_r/B_{r-1})^{p+r,q-r+1}$$

Lemma 4. Under the isomorphism $f\colon E_2^{p,q} \to \bar{E}_2^{p,q}$ )which holds for $p \leq P$) $B_r$ corresponds to $\bar{B}_r$ and $Z_r$ corresponds to $\bar{Z}_r$ for $p + r \leq P$.

Proof: Again by induction. For $r = 1$ our conventions make it trivial. For $r = 2$ it is also clear. The inductive step is made by inspecting the following diagram in which $p \leq P$.

$$\begin{array}{ccccccc}
(Z_{r-1}/B_{r-1})^{p-r,q+r-1} \cong E_r^{p-r,q+r-1} & \xrightarrow{d_r} & E_r^{p,q} & \xrightarrow{\text{mono}} & E_2^{p,q}/B_{r-1}^{p,q} \\
\cong \downarrow f \qquad\qquad \downarrow f & & \downarrow f & & f \downarrow \cong \\
(\bar{Z}_{r-1}/\bar{B}_{r-1})^{p-r,q+r-1} \xrightarrow{\cong} \bar{E}_r^{p-r,q+r-1} & \xrightarrow{d_r} & \bar{E}_r^{p,q} & \xrightarrow{\text{mono}} & \bar{E}_2^{p,q}/\bar{B}_{r-1}^{p,q}
\end{array}$$

Returning to the main line of argument, we now consider the group $E_2^{p,q}$ where $p + q = P \quad q \geq 1$. By the lemma $B_p$ $(= Z_{q+1})$ is preserved by $f$ and so is $Z_q$. Therefore $(Z_q/Z_{q+1})^{p,q}$ is mapped isomorphically by $f$. But

$$(Z_q/Z_{q+1})^{p,q} \xrightarrow[d_{q+1}]{\cong} (B_{q+1}/B_q)^{p+1,0} .$$

Therefore $(B_{q+1}/B_q)^{p+1,0}$ is mapped isomorphically by $f$ (for $1 \leq q \leq P$). Now $E_2^{P+1,0}$ has the composition series $0 = B_1 \subset B_2 \subset \ldots \subset B_{p+1} = Z_1 = E_2^{P+1,0}$. We have just shown that