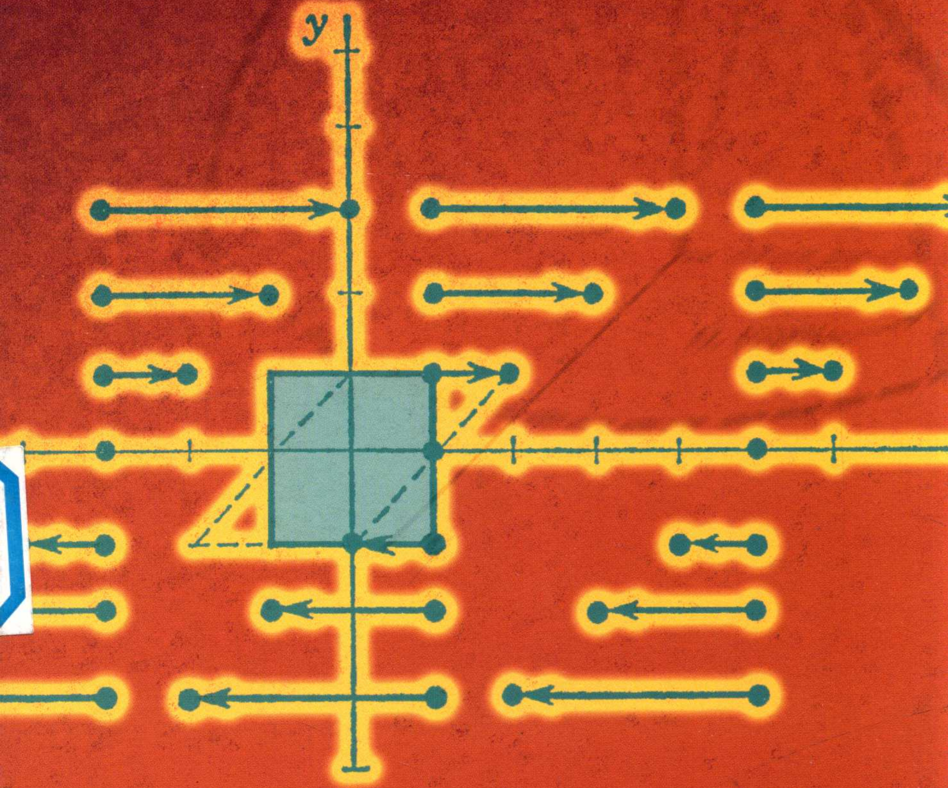
MATRIX VECTOR ANALYSIS

Richard L. Eisenman



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PREFACE

Vector analysis and linear algebra are often artificially separated. This book blends matrix algebra with vector analysis. Matrix ideas are applied to vector methods, and vector ideas are applied to matrix methods. The formal matrix product

$$(a, b, c)\begin{pmatrix}\bar{\imath}\\ \bar{\jmath}\\ \bar{k}\end{pmatrix} = a\bar{\imath} + b\bar{\jmath} + c\bar{k}$$

clarifies basis computations in Chap. I, derivatives with respect to moving bases in Chap. II, vector field theory in Chap. III, and vector operations in curvilinear coordinates in Chap. IV. Then row-vector interpretations clarify concrete applications of matrices to coding messages, algebraic equations, differential equations, linear transformations, eigenvalues, quadratic forms, and finite Markov chains in Chap. V. Matrix and vector concepts jointly lead to practical generalizations in group theory, linear spaces, function spaces, and n-dimensional spaces in Chap. VI. A blend of matrices and vectors is a natural preparation for tensor analysis. The excellent paperback "An Introduction to Tensor Analysis" by Professor H. D. Block is especially recommended.

The interplay of linear algebra and vector analysis is part of an attempt to dispel broader unnatural barriers between "pure" and "applied" mathematics. The attitude of this mathematics book has been influenced but not dictated by the constructive ideas of good engineers, including a formal opinion poll about applicable mathematics. Physical ideas are used freely to illustrate and motivate mathematical concepts, but the continuity of development is mathematical rather than physical. A fairly serious attempt is made to answer "What good is it?" while avoiding name-dropping.

The presentation is reasonably flexible. For an undergraduate three-semester-hour course in classical vector analysis, one might use Chaps. I through IV and either V or VI; perhaps both V and VI if the students have met vectors in elementary calculus. For an undergraduate linear-algebra course one might use Chaps. I, V, and VI and spend some time in creative laboratory sessions with the problem sets. Chapter I or Chap. V (except Sec. 5.5) might be used in the last semester of high school. The book may be useful as a supplement to formal courses or for review or self-study in advanced calculus, vector analysis, linear algebra, or tensor analysis.

Exercises are sprinkled within each section to encourage continuous understanding of fundamentals. Rather extensive problem sets at chapter ends demand enthusiasm but not genius. Mental exercises at chapter ends are intended for oral discussion or examination. At the end of the book the answer is given to each question for which a unique and short answer is possible. Such questions are indicated by bracketed numbers in the body of the book.

The encouragement and constructive criticism of M. E. Eisenman, R. C. Rounding, J. W. Ault, D. R. Barr, J. B. MacWherter, R. P. Yantis, W. Rollins, R. E. Thomas, N. Starr, and G. S. Young were requisite. It is clear that they are neither individually nor collectively responsible for shortcomings. Further criticism of this modest attempt to share ideas with growing minds will be most welcome.

Richard L. Eisenman

CONTENTS

chapter I | STATIC VECTORS IN 3 - SPACE

In this chapter you will enlarge your mathematical vocabulary by learning to manipulate in the arithmetic and algebra of vectors (as opposed to vector calculus).

1.1 | VECTORS AND EQUALITY

Professor J. Willard Gibbs of Yale published the first pamphlet on "Elements of Vector Analysis" about 15 years after the Civil War. The world was then ripe for a new language and new concepts to embody physical ideas. The notation of vector language has an inherent mechanical advantage, but you will profit even more from the vector concept than from the vector notation.

Young students are thoroughly indoctrinated in a narrow avenue of arithmetic elements in the real and complex numbers. The vector concept embodies elements of a more general nature, suitable for blending such global ideas as physical forces, solutions of algebraic equations, and spacial geometry to their mutual benefit.

You are invited to learn about vectors by doing.

The elements we call three-dimensional vectors are represented as directed line segments in usual three-dimensional space. The elementary geometry of length, angle, and line segment is presumed known and will provide the setting for the ideas to be explored in this applied vector space. A *directed* line segment simply has one end picked as tail and the other as head.

One and the same directed line segment or vector may represent the force of a boxer's punch, an arrow, a motion of space (from tail to tip) (it is in this context that the dictionary definition of a vector as a "carrier" makes sense), the point at the tip of the vector when its tail is bound at the origin, the trio of numbers $(0,1,2)$ which satisfies the algebraic equation $x + y + z = 3$, or the moment of a physical force. When you study this one vector, you are efficiently studying all these interpretations at the same time, focusing your attention on the properties of greatest concern which all the interpretations enjoy in common.

It is important to know how to distinguish between vectors which are "different" and to decide under what circumstances two vectors are indistinguishable or equal for our purposes. There are three important ways to classify vectors.

Definition: (Free) *vectors* are directed line segments which are called equal iff they have the same length and the same direction (iff means "if and only if"). A vector of zero length will be considered to have every direction, much as the number zero may be considered both positive and negative.

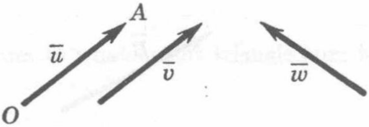
Bound vectors are directed line segments which are called equal iff they have the same length, direction, and starting point.

Sliding vectors are directed line segments which are called equal iff they have the same length and direction, and the heads and tails of both lie on a common line.

Conventions: *Vector elements will be denoted by letters with overscores; for example, $\bar{A}$, $\bar{V}$, $\bar{\phi}$, and $\bar{r}$ indicate vector elements. The symbol $\bar{\phi}$ will be reserved exclusively for the vector of zero length, and lowercase letters for vectors whose length is known to be one unit: so-called unit vectors.*

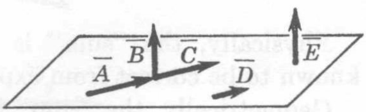
Other notations are bars, arrows, circumflexes, tildas, etc., over or under the letter. (Many texts simply use boldface type, which cannot be easily mimicked by the working student.) The student who carefully distinguishes between a vector and a scalar and never confuses the two earns an *A* in a vector course. (Is that grade a vector, or is it a scalar?)

Example:



As (free) *vectors:* $\overline{OA} = \bar{u} = \bar{v}$, but $\bar{v} \neq \bar{w}$.
As *bound vectors:* $\overline{OA} = \bar{u}$, but $\bar{u} \neq \bar{v}$, and $\bar{v} \neq \bar{w}$.

EXERCISE [1]: Which of the vectors in the diagram are equal?



a. As vectors (i.e., free vectors).
b. As bound vectors.
c. As sliding vectors.

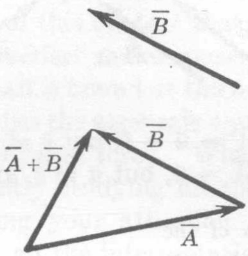
EXERCISE [2]: Which of the following have representations as vectors? **a.** Weight; **b.** Specific heat; **c.** Momentum; **d.** Energy; **e.** Speed; **f.** Velocity; **g.** Magnetic field intensity; **h.** Gravitational force; **i.** Kinetic energy; **j.** Age; **k.** Flux.

EXERCISE 3: Represent each of the following as a directed line segment:
a. The displacements of the moon from the center of the earth, from the Air Force Academy, and from the center of the sun (all in one figure)
b. A force of 1,000 lb on a satellite toward the center of the earth

1.2 | VECTOR SUM AND VECTOR PRODUCT

Vectors are of little value until you put them to work combining with each other to form new vectors. It is important to remember that we are considering free vectors. When a vector is transported parallel with itself, it remains the same vector, much as you are the same person when you take a walk. Unlike you, however, a vector may not change its direction or its length without becoming a new object. One natural way to combine two vectors to produce a new one is motivated by the physical idea of resultant or "sum."

Definition: *Triangle sum.* The *sum* of two vectors $\bar{A}$ and $\bar{B}$, indicated by $\bar{A} + \bar{B}$, is the vector obtained by placing the tail of the second at the tip of the first and joining the tail of the first and the tip of the second.
Graphically:

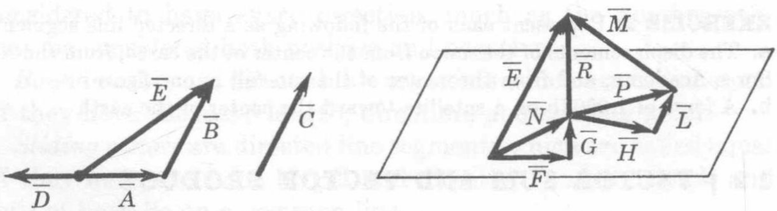


Physically, the "sum" is the resultant whose formation is known to be correct from experimental measures.

Geometrically, the "sum" is the shortcut from the tail of one to the tip of the other.

Algebraically, we will discover that this "sum" is governed by the same rules as the sum of numbers.

Example:



$\bar{A} + \bar{B} = \bar{E}$. And $\bar{A} + \bar{C} = \bar{A} + \bar{B}$, since $\bar{B}$ and $\bar{C}$ are the same vector.

$\bar{A} + \bar{D} = \bar{\phi}$, the vector of zero length. We write $\bar{D} = -\bar{A}$.

$\bar{N} = \bar{F} + \bar{G}$, $\bar{P} = \bar{H} + \bar{L}$, and $\bar{R} = \bar{P} + \bar{M}$. Therefore,

$$\bar{E} = \bar{N} + \bar{R} = (\bar{F} + \bar{G}) + [(\bar{H} + \bar{L}) + \bar{M}]$$

We may discard parentheses to write $\bar{E} = \bar{F} + \bar{G} + \bar{H} + \bar{L} + \bar{M}$.

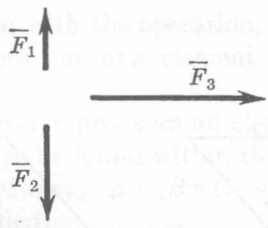
EXERCISE 4: From the definition of sum, all groupings by parentheses of a sum $\bar{A} + \bar{B} + \bar{C} + \bar{D}$ are equal. For example, $\bar{A} + [\bar{B} + (\bar{C} + \bar{D})] = (\bar{A} + \bar{B}) + (\bar{C} + \bar{D})$. How many such groupings are possible?

EXERCISE 5: Another definition of the vector sum of $\bar{A}$ and $\bar{B}$ is the directed diagonal of a parallelogram, two of whose sides are $\bar{A}$ and $\bar{B}$ joined tail to tail. Show an example of this sum, and discuss whether or not this parallelogram addition gives a vector equal to that given by triangle addition.

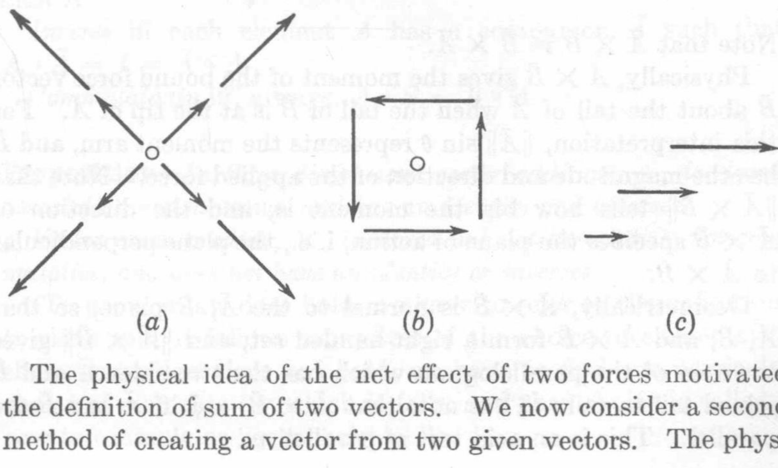
EXERCISE 6: Discuss to what extent triangle sum is defined for bound vectors.

EXERCISE 7: An airplane travels 200 miles north and then 100 miles 60° north of west. Determine the resultant displacement graphically.

EXERCISE 8: A satellite is acted upon by the forces shown. Determine the force needed to keep the satellite from moving.



EXERCISE 9: Determine the sums of the fields of vectors indicated:

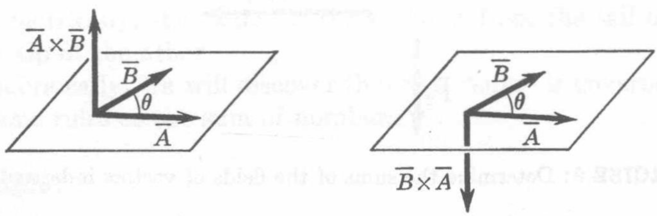


The physical idea of the net effect of two forces motivated the definition of sum of two vectors. We now consider a second method of creating a vector from two given vectors. The physical idea of moment about a point of a force whose tail has a displacement from that point motivates the definition of cross product or vector product.

Definitions: The *cross product*, denoted by $\bar{A} \times \bar{B}$, is the vector which protrudes perpendicular to the $\bar{A}$, $\bar{B}$ plane on the side that a right-handed screw would protrude if turned through the smaller angle from $\bar{A}$ toward $\bar{B}$; $\bar{A} \times \bar{B}$ has as its length $\|\bar{A}\| \|\bar{B}\| \sin \theta$, where θ is the angle between $\bar{A}$ and $\bar{B}$. Here $\|\bar{A}\|$ denotes the length of $\bar{A}$, and the *angle between* $\bar{A}$ and $\bar{B}$ is the smaller angle (or π) when the free vectors are joined tail to tail.

An ordered set of three vectors $\{\bar{A}, \bar{B}, \bar{C}\}$ is called a *right-handed set* if $\bar{C}$ protrudes from the $\bar{A}$, $\bar{B}$ plane on the same side as a right-handed screw turned through the angle between $\bar{A}$ and $\bar{B}$.

Example:



Note that $\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$.

Physically, $\bar{A} \times \bar{B}$ gives the moment of the bound force vector $\bar{B}$ about the tail of $\bar{A}$ when the tail of $\bar{B}$ is at the tip of $\bar{A}$. For this interpretation, $\|\bar{A}\| \sin \theta$ represents the moment arm, and $\bar{B}$ has the magnitude and direction of the applied force. Note that $\|\bar{A} \times \bar{B}\|$ tells how big the moment is, and the direction of $\bar{A} \times \bar{B}$ specifies the plane of action, i. e., the plane perpendicular to $\bar{A} \times \bar{B}$.

Geometrically, $\bar{A} \times \bar{B}$ is normal to the $\bar{A}$, $\bar{B}$ plane, so that $\bar{A}$, $\bar{B}$, and $\bar{A} \times \bar{B}$ form a right-handed set, and $\|\bar{A} \times \bar{B}\|$ gives the area of the parallelogram which has the two sides $\bar{A}$ and $\bar{B}$ joined tail to tail. Thus note that $\bar{A} \times \bar{B} = \bar{\phi}$ iff $\bar{A}$ and $\bar{B}$ are parallel. This is an acid test of parallelism.

It is important to realize that $\bar{A} \times \bar{B}$ contains a wealth of information. For example, once it is known that $\bar{A} \times \bar{B} = 1.1\bar{\jmath}$, where $\bar{\jmath}$ is a unit vector extending perpendicularly upward from the xz plane, one can conclude that:

1. $\bar{A}$ and $\bar{B}$ are parallel to the xz plane. (Note how the direction of $\bar{A} \times \bar{B}$ specifies the plane formed by $\bar{A}$ and $\bar{B}$.)
2. The parallelogram formed by $\bar{A}$ and $\bar{B}$ has area 1.1.

Algebraically, although $\bar{A} \times \bar{B}$ looks like a product, it does not obey all the laws of products of real numbers. For this reason, it is very important to examine those fundamental laws which each of the vector operations + and $\times$ do obey.

There are certain accepted names for the basic computational laws which are enjoyed by some (but not all) operations:

Definition: A system with the operation, $\circ$, for combining pairs of elements to produce one new element is said to satisfy the law of:

Closure iff $A \circ B$ always produces an element in the collection (An answer can always be found within the system)

Associativity iff, always, $A \circ (B \circ C) = (A \circ B) \circ C$ (Parentheses may be discarded)

Identity iff there is an I such that $A \circ I = I \circ A = A$ for each A

Inverse iff each element A has a companion $\tilde{A}$ such that $A \circ \tilde{A} = I = \tilde{A} \circ A$

Commutativity iff, always, $A \circ B = B \circ A$

Proposition: *In three dimensions vector addition, +, is closed, associative, commutative, and has an identity and inverses.*

Vector cross product, $\times$, is closed but not associative, not commutative, and does not have an identity or inverses.

(To prove a law does hold, we have to refer to the definitions to show that it follows regardless of the choice of elements; to show that a law does not hold, we have to find one particular choice of elements for which it fails, and that choice is called a counterexample or "gegenbeispiel.")

Proof: Addition is closed, because two free vectors may always be placed tail to head, and there is then a unique arrow sum.

Addition is associative, i.e., $\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$, because from the definition of triangle addition each expression gives the segment from the tail of $\bar{A}$ to the tip of $\bar{C}$.

Addition is commutative, for by properties of parallel lines the segment from the tail of $\bar{A}$ to the tip of $\bar{B}$ has the same direction and magnitude as the segment from the tail of $\bar{B}$ to the tip of $\bar{A}$. (Construct a figure for yourself.)

Addition has an identity, namely, the null or zero vector $(\bar{A} + \bar{\phi} = \bar{\phi} + \bar{A} = \bar{A})$.

Addition has an inverse for each $\bar{A}$, namely, $-\bar{A}$.

$$\bar{A} + (-\bar{A}) = (-\bar{A}) + \bar{A} = \bar{\phi}$$

Cross product is closed, for there is a unique perpendicular to the $\bar{A}$, $\bar{B}$ plane having the prescribed length and direction.

Cross product is not associative. As a counterexample, take $\bar{A}$ and $\bar{B}$ at a 30° angle in the plane of this page and $\bar{C}$ protruding from the page at 90°. Then $\bar{A} \times (\bar{B} \times \bar{C})$ will be perpendicular to this page, since $\bar{B} \times \bar{C}$ is on the page; but $(\bar{A} \times \bar{B}) \times \bar{C}$ will be $\bar{\phi}$ since $\bar{A} \times \bar{B}$ is perpendicular to this page and thus parallel to $\bar{C}$. (Of course this is not the only possible counterexample, but one failure is enough to ruin the general rule.)

Cross product is not commutative, for if nonparallel $\bar{A}$ and $\bar{B}$ are on this page, $\bar{A} \times \bar{B}$ will protrude from one side, while $\bar{B} \times \bar{A}$ protrudes from the other. Note, however, that cross product is anticommutative, that is, $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$.

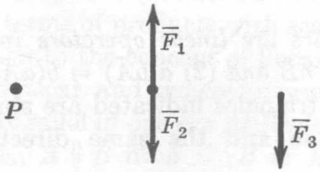
Cross product does not have an identity $\bar{I}$ such that

$$\bar{A} \times \bar{I} = \bar{A}$$

for if it did, $\bar{A}$ would have to be perpendicular to itself. Without an identity inverses do not make sense as we defined them.

EXERCISE [10]: If $\{\bar{u}, \bar{v}, \bar{w}\}$ is a right-handed set, which of the following sets is right-handed: $\bar{u}, \bar{w}, \bar{v}$; $\bar{v}, \bar{w}, \bar{u}$; $\bar{v}, \bar{u}, \bar{w}$; $\bar{w}, \bar{u}, \bar{v}$; $\bar{w}, \bar{v}, \bar{u}$?

EXERCISE 11: Show the moment of each force in the diagram about the point P.



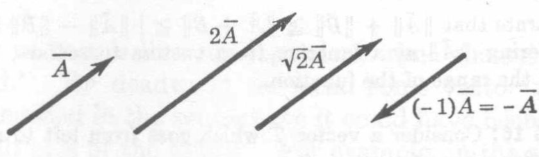
EXERCISE 12: Write at least five identities which hold if the elements are real numbers and "×" stands for multiplication, but which do not hold if the elements are vectors and "×" stands for vector cross product.

1.3 | SCALAR MULTIPLICATION AND LINEAR DEPENDENCE

As a special case of sum, $\bar{A} + \bar{A}$ is a new vector which is naturally called $2\bar{A}$. More generally, we can always alter the length of a vector while retaining its direction, simply by multiplying the original length by some real number. This process is called scalar multiplication.

Definition: The *scalar multiple* $a\bar{A}$ is a vector parallel to $\bar{A}$, has length $|a|\,\|\bar{A}\|$, and is directed along or opposite $\bar{A}$ according as a is a positive or a negative real number.

Example:



Since $\bar{A} + (-1)\bar{A} = \bar{\phi}$, we have a natural definition of subtraction:

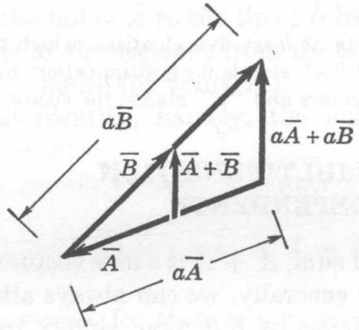
Definition: $\bar{A} - \bar{B} = \bar{A} + (-1)\bar{B}$

Scalar multiplication is not quite like multiplication of numbers, because two different kinds of objects are involved. A scalar (real number) operates on a vector to produce a new vector.

Thus scalars, which are not elements in the space of vectors, do *operate* on vectors, and one speaks of "the scalar operators."

Proposition: *Scalars are linear operators in the sense that* (1) $a(\bar{A} + \bar{B}) = a\bar{A} + a\bar{B}$ *and* (2) $a(b\bar{A}) = b(a\bar{A})$.

Proof: Since the triangles indicated are similar, $a\bar{A} + a\bar{B}$ has the same magnitude and the same direction as $a(\bar{A} + \bar{B})$.



Property 2 follows from the definition.

EXERCISE 13: *Prove:* $(a + b)\bar{A} = a\bar{A} + b\bar{A}$. Do the signs "+" refer to adding numbers or to adding vectors?

EXERCISE [14]: Discuss the significance of $\|\bar{A}\|^{-1}\bar{A}$. (What is its length?)

EXERCISE 15: $\|\bar{A}\|$ behaves somewhat like the absolute value of a real number.
a. Demonstrate that $\|\bar{A}\| + \|\bar{B}\| \geqq \|\bar{A} + \bar{B}\| \geqq |\ \|\bar{A}\| - \|\bar{B}\|\ |$.
[b] Considering $\|\bar{A}\|$ as a function from vectors to scalars, describe the domain and the range of the function.

EXERCISE 16: Consider a vector $\bar{T}$ which goes from left to right on the top edge of this piece of paper.
a. Describe $-\bar{T}$ verbally.
b. In order not to destroy this free vector, how would you have to carry this book?
[c] Which of the vectors—from left to middle or from middle to right—is equal to $\frac{1}{2}\bar{T}$?

EXERCISE [17]: Write an algebraic expression which describes every vector parallel to the vector $\bar{k}$.