Graduate Texts in Mathematics

Klaus Fritzsche
Hans Grauert

From Holomorphic Functions to Complex Manifolds

从全纯函数到复流形

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Klaus Fritzsche
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(continued after index)

Preface

The aim of this book is to give an understandable introduction to the theory of complex manifolds. With very few exceptions we give complete proofs. Many examples and figures along with quite a few exercises are included. Our intent is to familiarize the reader with the most important branches and methods in complex analysis of several variables and to do this as simply as possible. Therefore, the abstract concepts involved with sheaves, coherence, and higher-dimensional cohomology are avoided. Only elementary methods such as power series, holomorphic vector bundles, and one-dimensional cocycles are used. Nevertheless, deep results can be proved, for example the Remmert–Stein theorem for analytic sets, finiteness theorems for spaces of cross sections in holomorphic vector bundles, and the solution of the Levi problem.

The first chapter deals with holomorphic functions defined in open subsets of the space $\mathbb{C}^n$. Many of the well-known properties of holomorphic functions of one variable, such as the Cauchy integral formula or the maximum principle, can be applied directly to obtain corresponding properties of holomorphic functions of several variables. Furthermore, certain properties of differentiable functions of several variables, such as the implicit and inverse function theorems, extend easily to holomorphic functions.

In Chapter II the following phenomenon is considered: For $n \geq 2$, there are pairs of open subsets $H \subset P \subset \mathbb{C}^n$ such that every function holomorphic in H extends to a holomorphic function in P. Special emphasis is put on domains $G \subset \mathbb{C}^n$ for which there is no such extension to a bigger domain. They are called domains of holomorphy and have a number of interesting convexity properties. These are described using plurisubharmonic functions. If G is not a domain of holomorphy, one asks for a maximal set E to which all holomorphic functions in G extend. Such an "envelope of holomorphy" exists in the category of Riemann domains, i.e., unbranched domains over $\mathbb{C}^n$.

The common zero locus of a system of holomorphic functions is called an analytic set. In Chapter III we use Weierstrass's division theorem for power series to investigate the local and global structure of analytic sets. Two of the main results are the decomposition of analytic sets into irreducible components and the extension theorem of Remmert and Stein. This is the only place in the book where singularities play an essential role.

Chapter IV establishes the theory of complex manifolds and holomorphic fiber bundles. Numerous examples are given, in particular branched and unbranched coverings of $\mathbb{C}^n$, quotient manifolds such as tori and Hopf manifolds, projective spaces and Grassmannians, algebraic manifolds, modifications, and toric varieties. We do not present the abstract theory of complex spaces, but do provide an elementary introduction to complex algebraic geometry. For example, we prove the theorem of Chow and we cover the theory of divisors and hyperplane sections as well as the process of blowing up points and submanifolds.

The present book grew out of the old book of the authors with the title *Several Complex Variables*, Graduate Texts in Mathematics 38, Springer Heidelberg, 1976. Some of the results in Chapters I, II, III, and V of the old book can be found in the first four chapters of the new one. However, these chapters have been substantially rewritten. Sections on pseudoconvexity and on the structure of analytic sets; the entire theory of bundles, divisors, and meromorphic functions; and a number of examples of complex manifolds have been added.

Our exposition of Stein theory in Chapter V is completely new. Using only power series, some geometry, and the solution of Cousin problems, we prove finiteness and vanishing theorems for certain one-dimensional cohomology groups. Neither sheaf theory nor $\overline{\partial}$ methods are required. As an application Levi's problem is solved. In particular, we show that every pseudoconvex domain in $\mathbb{C}^n$ is a domain of holomorphy.

Through Chapter V we develop everything in full detail. In the last two chapters we deviate a bit from this principle. Toward the end, a number of the results are only sketched. We do carefully define differential forms, higher-dimensional Dolbeault and de Rham cohomology, and Kähler metrics. Using results of the previous sections we show that every compact complex manifold with a positive line bundle has a natural projective algebraic structure. A consequence is the algebraicity of Hodge manifolds, from which the classical period relations are derived. We give a short introduction to elliptic operators, Serre duality, and Hodge and Kodaira decomposition of the Dolbeault cohomology. In such a way we present much of the material from complex differential geometry. This is thought as a preparation for studying the work of Kobayashi and the papers of Ohsawa on pseudoconvex manifolds.

In the last chapter real methods and recent developments in complex analysis that use the techniques of real analysis are considered. Kähler theory is carried over to strongly pseudoconvex subdomains of complex manifolds. We give an introduction to Sobolev space theory, report on results obtained by J.J. Kohn, Diederich, Fornæss, Catlin, and Fefferman ($\overline{\partial}$-Neumann, subelliptic estimates), and sketch an application of harmonic forms to pseudoconvex domains containing nontrivial compact analytic subsets. The Kobayashi metric and the Bergman metric are introduced, and theorems on the boundary behavior of biholomorphic maps are added.

Prerequisites for reading this book are only a basic knowledge of calculus, analytic geometry, and the theory of functions of one complex variable, as well as a few elements from algebra and general topology. Some knowledge about Riemann surfaces would be useful, but is not really necessary. The book is written as an introduction and should be of interest to the specialist and the nonspecialist alike.

We are indebted to many colleagues for valuable suggestions, in particular to K. Diederich, who gave us his view of the state of the art in $\overline{\partial}$-Neumann theory. Special thanks go to A. Huckleberry, who read the manuscript with great care and corrected many inaccuracies. He made numerous helpful suggestions concerning the mathematical content as well as our use of the English language. Finally, we are very grateful to the staff of Springer-Verlag for their help during the preparation of our manuscript.

Wuppertal, Göttingen, Germany
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Klaus Fritzsche
Hans Grauert

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Chapter I

Holomorphic Functions

1. Complex Geometry

Real and Complex Structures. Let V be an n-dimensional complex vector space. Then V can also be regarded as a $2n$-dimensional real vector space, and multiplication by $\mathrm{i} := \sqrt{-1}$ gives a real endomorphism $\mathsf{J} : V \to V$ with $\mathsf{J}^2 = -\mathrm{id}_V$. If $\{a_1, \dots, a_n\}$ is a complex basis of V, then $\{a_1, \dots, a_n, \mathrm{i}a_1, \dots, \mathrm{i}a_n\}$ is a real basis of V.

On the other hand, given a $2n$-dimensional real vector space V, every real endomorphism $\mathsf{J} : V \to V$ with $\mathsf{J}^2 = -\mathrm{id}_V$ induces a *complex structure* on V by

$$(a + \mathrm{i}b) \cdot v := a \cdot v + b \cdot \mathsf{J}(v).$$

We denote this complex vector space also by V, or by (V, J), if we want to emphasize the complex structure.

If a complex structure J is given on V, then $-\mathsf{J}$ is also a complex structure. It is called the *conjugate complex structure*, and the space (V, J) is sometimes denoted by $\overline{V}$. A vector $v \in V$ is also a vector in $\overline{V}$. If z is a complex number, then the product $z \cdot v$, formed in $\overline{V}$, gives the same vector as the product $\overline{z} \cdot v$ in V.

Our most important example is the *complex n-space*

$$\mathbb{C}^n := \{\mathbf{z} := (z_1, \dots, z_n) \,:\, z_i \in \mathbb{C} \text{ for } i = 1, \dots, n\},$$

with the *standard basis*

$$\mathbf{e}_1 := (1, 0, \dots, 0), \; \dots \;, \mathbf{e}_n := (0, \dots, 0, 1).$$

We can interpret $\mathbb{C}^n$ as the *real $2n$-space*

$$\mathbb{R}^{2n} = \{(\mathbf{x}, \mathbf{y}) = (x_1, \dots, x_n, y_1, \dots, y_n) \,:\, x_i, y_i \in \mathbb{R} \text{ for } i = 1, \dots, n\}^1,$$

together with the complex structure $\mathsf{J} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, given by

$$\mathsf{J}(x_1, \dots, x_n, y_1, \dots, y_n) := (-y_1, \dots, -y_n, x_1, \dots, x_n).$$

These considerations lead naturally to the idea of "complexification."

[1] A row vector is described by a bold symbol, for instance $\mathbf{v}$, whereas the corresponding column vector is written as a transposed vector: $\mathbf{v}^t$.

Definition. Let E be an n-dimensional real vector space. The *complexification* of E is the real vector space $E_c := E \oplus E$, together with the complex structure $\mathsf{J} : E_c \to E_c$, given by

$$\mathsf{J}(v, w) := (-w, v).$$

Furthermore, *conjugation* C in E_c is defined by

$$\mathsf{C}(v, w) := (v, -w).$$

Since $\mathsf{C} \circ \mathsf{J} = -\mathsf{J} \circ \mathsf{C}$, it is clear that C defines a complex isomorphism between E_c and $\overline{E_c}$.

The complexification of $\mathbb{R}^n$ is the complex n-space $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$ in the way shown above. In this case the conjugation C is given by

$$\mathsf{C} : (z_1, \dots, z_n) \mapsto (\overline{z}_1, \dots, \overline{z}_n)$$

and will also be denoted by $\mathbf{z} \mapsto \overline{\mathbf{z}}$.

If $V = E_c$ is the complexification of a real vector space E, then the subspace

$$\mathrm{Re}(V) := \{(v, 0) : v \in E\} \subset V$$

is called the *real part* of V. Since it is isomorphic to E in a natural way, we can write $V \cong E \oplus \mathrm{i}E$. If V is an arbitrary complex vector space, then V is the complexification of some real vector space as well, but this real part is not uniquely defined. It is given by the real span of any complex basis of V.

Example

Let E be an n-dimensional real vector space and $E^* := \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ the real dual space of linear forms on E. Then the complexification $(E^*)_c$ can be identified with the space $\mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})$ of complex-valued linear forms on E.

In the case $E = \mathbb{R}^n$, a linear form $\lambda \in E^*$ is always given by

$$\lambda : \mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{a}^t,$$

with some fixed vector $\mathbf{a} \in \mathbb{R}^n$. An element of the complexification $(E^*)_c$ is then given by $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{z}^t$ with $\mathbf{z} = \mathbf{a} + \mathrm{i}\mathbf{b} \in (\mathbb{R}^n)_c = \mathbb{C}^n$.

Now let T be an n-dimensional *complex* vector space and $F(T) := \mathrm{Hom}_{\mathbb{R}}(T, \mathbb{C})$ the space of complex-valued *real* linear forms on T. It contains the subspaces $T' := \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$ of complex linear forms and $\overline{T}' := \mathrm{Hom}_{\mathbb{C}}(\overline{T}, \mathbb{C})$ of complex antilinear forms [2].

[2] A real linear map $\lambda : T \to \mathbb{C}$ is called *complex antilinear* if $\lambda(c \cdot v) = \overline{c} \cdot \lambda(v)$ for $c \in \mathbb{C}$. Therefore, $\overline{T}'$ can be viewed as the set of complex antilinear forms on T.