Controllability and Observability for Quasilinear Hyperbolic Systems

拟线性双曲系统的能控性与能观性

Tatsien Li

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Preface

The controllability and observability are of great importance in both theory and applications. A complete theory has been established for linear hyperbolic systems, in particular, for linear wave equations. There have also been some results for semilinear wave equations. For quasilinear hyperbolic systems, however, very few results have been published even in the one-space-dimensional (1-D) case.

In this monograph based mainly on the results obtained by the author and his collaborators in recent years, by means of the theory on the semi-global classical solution, a simple and direct constructive method is presented in a systematic way to get both the controllability and observability in the framework of classical solutions for general first order 1-D quasilinear hyperbolic systems with general nonlinear boundary conditions, and corresponding applications are given for 1-D quasilinear wave equations and for unsteady flows in a tree-like network of open canals, respectively. This will be of benefit to scholars and graduate students in applied mathematics and in applied sciences.

The Appendix given at the end of this monograph is specially written for those readers who are not familiar with quasilinear hyperbolic systems.

I would like to take this opportunity to express my sincere thanks to the late professor J.-L. Lions, who initiated and brought me into the area of control theory, for his encouragement and guidance. My special thanks are due to Bopeng Rao, Binyu Zhang, Yi Jin, Lixin Yu, Zhiqiang Wang and Qilong Gu for their kind cooperation in the course of research on this subject, supported by the National Basic Research Program of China (973 Program) (2007CB814800). Finally, I am also indebted to Ms. Chunlian Zhou for her patient and efficient work in editing this book.

October 2009

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1

Introduction

1.1 Exact Controllability

What is the exact controllability? Let us begin from the simplest situation. Consider the following system of linear ODEs

$$\frac{dX}{dt} = AX + Bu, \tag{1.1}$$

where t is the independent variable (time), $X = (X_1, \cdots, X_N)$ is the state variable, $u = (u_1, \cdots, u_m)$ is the control variable, A and B are $N \times N$ and $N \times m$ constant matrices respectively.

This system possesses the **exact controllability** on the interval $[0, T]$ ($T > 0$), if, for any given initial data X_0 at $t = 0$ and any given final data X_T at $t = T$, we can find a control function $u = u(t)$ on $[0, T]$, such that the solution $X = X(t)$ to the Cauchy problem

$$\begin{cases} \dfrac{dX}{dt} = AX + Bu(t), & (1.2) \\ t = 0: \quad X = X_0 & (1.3) \end{cases}$$

verifies exactly the final condition

$$t = T: \quad X = X_T. \tag{1.4}$$

It is well-known that system (1.1) possesses the exact controllability on $[0, T]$, if and only if the matrix

$$[B \vdots AB \vdots \cdots \vdots A^{N-1}B] \tag{1.5}$$

is full-rank (cf. [77]). Hence, if system (1.1) is exactly controllable on an interval $[0, T]$ ($T > 0$), then it is also exactly controllable on any interval $[0, T_1]$ ($T_1 > 0$), in particular, the exact controllability can be realized almost immediately.

We now consider the exact controllability for hyperbolic systems of PDEs. For this purpose, several points different from the ODE case should be pointed out as follows.

1. In order to solve a hyperbolic system on a bounded domain (or on a domain with boundary), one should prescribe suitable boundary conditions. As a result, the control may be an **internal control** appearing in the equation like in the ODE case and acting on the whole domain or a part of domain, or a **boundary control** appearing in the boundary conditions and acting on the whole boundary or a part of boundary.

Since the boundary control is much easier than the internal control to be handled in practice, we concentrate our attention mainly on the **exact boundary controllability**, namely, the exact controllability realized only by boundary controls.

The **exact boundary controllability** means that there exists $T > 0$ such that by means of boundary controls, the system (hyperbolic equations together with boundary conditions) can drive any given initial data at $t = 0$ to any given final data at $t = T$.

If the exact boundary controllability can be realized only for small (in some sense!) initial data and final data, it is called to be a **local exact boundary controllability**; otherwise, a **global exact boundary controllability**.

2. Since the hyperbolic wave has a finite speed of propagation, the exact boundary controllability time $T > 0$ should be suitably large.

In fact, for any given initial data, by solving the corresponding **forward** Cauchy problem, there is a unique solution on its maximum determinate domain.

Similarly, for any given final data, by solving the corresponding **backward** Cauchy problem, there is a unique solution on its maximum determinate domain.

In order to ensure the consistency, these two maximum determinate domains should not intersect each other (Figure 1.1), then $T(> 0)$ must be suitably large.

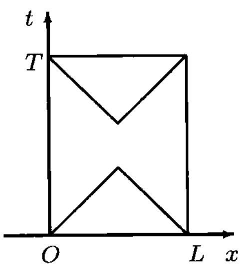

Figure 1.1

On the other hand, from the point of view of applications, $T(>0)$ should be chosen as small as possible.

3. For the weak solution to quasilinear hyperbolic systems, which includes shock waves and corresponds to an irreversible process, generically speaking, it is impossible to have the exact boundary controllability for any arbitrarily given initial and final states (cf. [7]). Of course, by requiring certain additional restrictions on the initial state and the final state (particularly on the later) and, perhaps, suitably weakening the definition of controllability, it is still possible to consider the exact boundary controllability in the framework of weak solutions, however, up to now several results obtained in this direction with different methods are only for very special quasilinear hyperbolic systems (the scalar convex conservation law [3–4], [32], genuinely nonlinear systems of Temple class [2] and the p-system in isentropic gas dynamics [17]). Hence, in order to give a general and systematic presentation, in this book we restrict ourselves to the consideration in the framework of classical solutions, namely, the solution under consideration to the hyperbolic system means its classical solution which corresponds to a reversible process.

We know that for nonlinear hyperbolic problems, there is always the local existence and uniqueness of classical solutions, provided that the initial data and the boundary data are smooth and suitable conditions of compatibility hold; but, generically speaking, the classical solution exists only locally in time (see [33–34], [39], [41]). However, as we said before, in order to guarantee the exact boundary controllability, we should have a classical solution on the interval $[0,T]$, where $T>0$ might be suitably large. This kind of classical solution is called to be a **semi-global classical solution** (see Chapter 2), which is different from either the local classical solution or the global classical solution (cf. [10], [53], [60–62]).

Thus, the existence of semi-global classical solution is an important basis for the exact boundary controllability.

Since, generically speaking, the semi-global classical solution to quasilinear hyperbolic systems exists only for small initial and boundary data and keeps small in its existence domain (see Chapter 2), in general one can only expect to have the local exact controllability in the quasilinear case. However, it is still possible to get the global exact controllability in some special cases (see Remark 3.9).

In the case of hyperbolic PDEs, most studies on the controllability are concentrated on the wave equation

$$u_{tt} - \triangle u = 0 \tag{1.6}$$

(cf. [73–75] and the references therein). Moreover, there are some results for semilinear wave equations

$$u_{tt} - \triangle u = F(u) \tag{1.7}$$

(cf. [15–16], [35], [99–100], [103]). However, in the quasilinear case, very few results have been published even for the 1-D quasilinear hyperbolic PDEs (see [9]).

In this book we shall consider the exact boundary controllability for first order quasilinear hyperbolic systems with general nonlinear boundary conditions in one-space dimensional case.

More precisely, we consider the following first order 1-D quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.8}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x), $A(u)$ is an $n \times n$ matrix with smooth entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, $F(u) = (f_1(u), \cdots, f_n(u))^T$ is a smooth vector function of u with

$$F(0) = 0. \tag{1.9}$$

Obviously, $u = 0$ is an **equilibrium** of (1.8).

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete set of left eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ $(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u). \tag{1.10}$$

In particular, when $A(u)$ has n distinct real eigenvalues

$$\lambda_i(u) < \lambda_2(u) < \cdots < \lambda_n(u) \tag{1.11}$$

on the domain under consideration, system (1.8) is called to be **strictly hyperbolic**.

Suppose that there are no zero eigenvalues:

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \cdots, m;\ s = m + 1, \cdots, n). \tag{1.12}$$

In this situation, the subscripts $r = 1, \cdots, m$ (resp. $s = m + 1, \cdots, n$) are always used to correspond to the negative (resp. positive) eigenvalues.

Let

$$v_i = l_i(u)u \qquad (i = 1, \cdots, n). \tag{1.13}$$

v_i is called to be the **diagonal variable** corresponding to the i-th eigenvalue $\lambda_i(u)$.

The boundary conditions are given by

$$x = 0: \quad v_s = G_s(t, v_1, \cdots, v_m) + H_s(t) \tag{1.14}$$
$$(s = m + 1, \cdots, n),$$
$$x = L: \quad v_r = G_r(t, v_{m+1}, \cdots, v_n) + H_r(t) \tag{1.15}$$
$$(r = 1, \cdots, m),$$

where L is the length of the space interval $0 \leq x \leq L$, G_i $(i = 1, \cdots, n)$ are suitably smooth functions and, without loss of generality, we assume

$$G_i(t, 0, \cdots, 0) \equiv 0 \qquad (i = 1, \cdots, n); \tag{1.16}$$

moreover, all $H_i(t)(i = 1, \cdots, n)$ or a part of $H_i(t)(i = 1, \cdots, n)$ will be chosen as boundary controls.

(1.14)–(1.15) are the most general nonlinear boundary conditions to guarantee the well-posedness for the forward problem, the characters of which can be shown as

1) The number of boundary conditions on $x = 0$ (resp. on $x = L$) is equal to the number of positive (resp. negative) eigenvalues.

2) The boundary conditions on $x = 0$ (resp. on $x = L$) are written in the form that the diagonal variables v_s $(s = m+1, \cdots, n)$ corresponding to positive eigenvalues (resp. the diagonal variables v_r $(r = 1, \cdots, m)$ corresponding to negative eigenvalues) are explicitly expressed by the other diagonal variables.

For any given initial condition

$$t = 0: \quad u = \varphi(x), \quad 0 \leq x \leq L \tag{1.17}$$

and any given final condition

$$t = T: \quad u = \Phi(x), \quad 0 \leq x \leq L \tag{1.18}$$

with small C^1 norms $\|\varphi\|_{C^1[0,L]}$ and $\|\Phi\|_{C^1[0,L]}$, by means of the theory on the semi-global classical solution (see Chapter 2), we shall present a direct and simple constructive method to show the following theorems on the local exact boundary controllability (see Chapter 3).

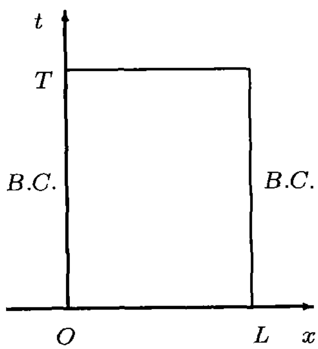

Figure 1.2

Theorem 1.1 (Two-sided control, [55]). *If*

$$T > L \max_{\substack{r=1,\cdots,m \\ s=m+1,\cdots,n}} \left(\frac{1}{|\lambda_r(0)|}, \frac{1}{\lambda_s(0)} \right), \tag{1.19}$$

then there exist boundary controls $H_i(t)$ $(i = 1, \cdots, n)$ with small $C^1[0,T]$ norm, such that the corresponding mixed initial-boundary value problem (1.8), (1.17) and (1.14)–(1.15) admits a unique semi-global C^1 solution $u = u(t,x)$ with small C^1 norm on the domain $R(T) = \{(t,x)|\ 0 \le t \le T,\quad 0 \le x \le L\}$, which verifies exactly the final condition (1.18) (Figure 1.2).

Theorem 1.2 (One-sided control, [54]). *Suppose that the number of positive eigenvalues is not bigger than that of negative ones:*

$$\bar{m} \stackrel{\text{def}}{=} n - m \le m, \quad i.e.,\ n \le 2m. \tag{1.20}$$

Suppose furthermore that boundary condition (1.14) on $x = 0$ can be equivalently rewritten in a neighbourhood of $u = 0$ as

$$x = 0: \quad v_{\bar{r}} = \bar{G}_{\bar{r}}(t, v_{\bar{m}+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) + \bar{H}_{\bar{r}}(t) \\ (\bar{r} = 1, \cdots, \bar{m}), \tag{1.21}$$

where

$$\bar{G}_{\bar{r}}(t, 0, \cdots, 0) \equiv 0 \qquad (\bar{r} = 1, \cdots, \bar{m}), \tag{1.22}$$

then

$$\|\bar{H}_{\bar{r}}\|_{C^1[0,T]}\ (\bar{r} = 1, \cdots, \bar{m}) \text{ small enough} \\ \Longleftrightarrow \|H_s\|_{C^1[0,T]}\ (s = m+1, \cdots, n) \text{ small enough.} \tag{1.23}$$

If

$$T > L\left(\max_{r=1,\cdots,m} \frac{1}{|\lambda_r(0)|} + \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(0)} \right), \tag{1.24}$$

then, for any given $H_s(t)$ $(s = m+1, \cdots, n)$ with small $C^1[0,T]$ norm, satisfying the conditions of C^1 compatibility at the points $(t,x) = (0,0)$ and

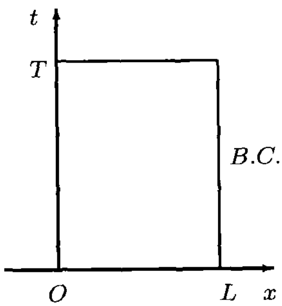

Figure 1.3

$(T,0)$ respectively, there exist boundary controls $H_r(t)$ $(r = 1,\cdots,m)$ at $x = L$ with small $C^1[0,T]$ norm, such that the corresponding mixed initial-boundary value problem (1.8), (1.17) and (1.14)–(1.15) admits a unique semi-global C^1 solution $u = u(t,x)$ with small C^1 norm on the domain $R(T) = \{(t,x)|\ 0 \le t \le T,\quad 0 \le x \le L\}$, which verifies exactly the final condition (1.18) (Figure 1.3).

Theorem 1.3 (Two-sided control with less controls, [96]). *Suppose that the number of positive eigenvalues is less than that of negative ones:*

$$\bar{m} \overset{\text{def}}{=} n - m < m, \quad i.e.,\ n < 2m. \tag{1.25}$$

Suppose furthermore that, without loss of generality, the first $\bar{m}$ boundary conditions in (1.15) at $x = L$, namely,

$$x = L: \quad v_{\bar{r}} = G_{\bar{r}}(t, v_{m+1},\cdots,v_n) + H_{\bar{r}}(t) \quad (\bar{r} = 1,\cdots,\bar{m}), \tag{1.26}$$

can be equivalently rewritten in a neighbourhood of $u = 0$ as

$$x = L: \quad v_s = \bar{G}_s(t, v_1,\cdots,v_{\bar{m}}) + \bar{H}_s(t) \quad (s = m+1,\cdots,n), \tag{1.27}$$

where

$$\bar{G}_s(t,0,\cdots,0) \equiv 0 \quad (s = m+1,\cdots,n), \tag{1.28}$$

then

$$\begin{aligned} &\|\bar{H}_s\|_{C^1[0,T]}\ (s = m+1,\cdots,n) \text{ small enough} \\ \Longleftrightarrow &\|H_{\bar{r}}\|_{C^1[0,T]}\ (\bar{r} = 1,\cdots,\bar{m}) \text{ small enough.} \end{aligned} \tag{1.29}$$

If $T > 0$ satisfies (1.24), then, for any given $H_{\bar{r}}(t)$ $(\bar{r} = 1,\cdots,\bar{m})$ with small $C^1[0,T]$ norm, satisfying the conditions of C^1 compatibility at the points $(t,x) = (0,L)$ and (T,L) respectively, there exist boundary controls $H_s(t)$ $(s = m+1,\cdots,n)$ at $x = 0$ and $H_{\bar{\bar{r}}}(t)(\bar{\bar{r}} = \bar{m}+1,\cdots,m)$ at $x = L$ with small $C^1[0,T]$ norm, such that the corresponding mixed initial-boundary value problem (1.8), (1.17) and (1.14)–(1.15) admits a unique semi-global C^1 solution $u = u(t,x)$ with small C^1 norm on the domain $R(T) = \{(t,x)\Big|\ 0 \le t \le T,\ 0 \le x \le L\}$, which verifies exactly the final condition (1.18) (Figure 1.4).

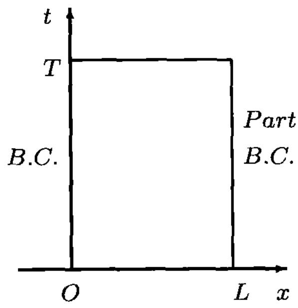

Figure 1.4

Remark 1.1. In the case of two-sided control, the number of boundary controls is equal to n, the number of unknown variables, namely, that of all the eigenvalues.

Remark 1.2. In the case of one-sided control, the number of boundary controls is reduced to the maximum value between the number of positive eigenvalues and the number of negative eigenvalues, and the boundary controls act only on the side with more boundary conditions, however, the controllability time must be enlarged.

In particular, when the number of positive eigenvalues is equal to the number of negative eigenvalues, boundary controls can act on each side.

Remark 1.3. In the case of two-sided control with less controls, both the number of boundary controls and the controllability time are as in the case of one-sided control, however, one needs all the boundary controls acting on the side with less boundary conditions and a part of boundary controls acting on the side with more boundary conditions.

Remark 1.4. The estimate on the exact controllability time T in Theorems 1.1–1.3 is sharp.

Remark 1.5. The boundary controls which realize the exact boundary controllability are not unique.

1.2 Exact Observability

Consider the system of linear ODEs

$$\frac{dX}{dt} = AX, \tag{1.30}$$

where $X = (X_1, \cdots, X_N)$ and A is an $N \times N$ constant matrix.

For any given initial data

$$t = 0: \quad X = X_0, \tag{1.31}$$

Cauchy problem (1.30)–(1.31) admits a unique solution $X = X(t)$.

Let

$$Y(t) = DX(t) \tag{1.32}$$

be the corresponding **observed value**, where D is an $m \times N$ constant matrix.

System (1.30) with (1.32) possesses the **exact observability** on the interval $[0, T]$ $(T > 0)$, if the observed value $Y(t)$ on the interval $[0, T]$ determines uniquely the initial data X_0 (then the solution $X(t)$ on any interval $[0, \tilde{T}]$).

It is well-known that system (1.30) with (1.32) possesses the exact observability on $[0, T]$, if and only if the matrix