Tazid Ali

Fuzzy Algebra

Some Advances

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Tazid Ali

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Dedicated to my
wife RUBAB and
son DAANISH

Preface

As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics.
L.A. Zadeh

When he introduced fuzzy sets forty five years ago, L. A. Zadeh (1965), was motivated by the lack of any existing mathematical framework that could cope with the complexity of *animate*, either biological or humanistic, systems. Since then there has been a tremendous interest in the subject. Now fuzzy sets have advanced in a variety of ways and in many disciplines. Almost all mathematical, engineering, medicinal etc. concepts have been redefined using fuzzy sets. Applications of this theory can be found in artificial intelligence, computer science, control; engineering, decision theory, expert systems, logic, management science, operation research, pattern recognition, robotics besides classical mathematics areas such as algebra, topology, graph theory. Hence it is necessary to popularize these ideas for our future generation.

The objective of this monograph is to bring together some recent developments in the field of fuzzification of algebraic structures like groups, rings, vector spaces and lattices. The monograph is divided into six chapters. The first chapter discusses the various fundamental aspects of fuzzy set theory. In chapter 2 we deal with various aspects of fuzzy functions between groups, rings and vector spaces. We have characterized fuzzy homomorphism of groups and dealt with morphisms based on fuzzy point. In chapter 3 we have studied some algebraic structures generated by translation invariant fuzzy subsets. Chapter 4 is devoted to the study of some L-sub-structures of groups. In chapter 5 we have developed different types of fuzzy substructures in a lattice by defining two binary operations o and $*$ on the set of fuzzy subsets. The last chapter deals with product structures of different fuzzy algebraic structures.

The expected audience of this monograph are post graduate students and researchers in the field of fuzzy algebra. Though no backgrounds knowledge of fuzzy mathematics is assumed, the readers are expected to have a fair idea of the basics of algebraic structures like group, ring, vector space and lattice.

This monograph is a natural outgrowth of research works [1, 2, 3, 4, 5, 50, 51, 52, 53, 54, 55, 58, 59]. I would like to express my sincere thanks and gratefulness to Dr. A.K. Ray of the Department of Mathematics, Dibrugarh University for his constant support, inspiration and encouragement since I was his Ph.D student. We have co-authored many research papers. I am also thankful to Professor B.K. Sarma, Department of Mathematics, IIT Guwahati who initiated me to the field of fuzzy algebra and with whom I have co-authored two of my initial papers. Finally I would like to convey my thankfulness to Lap Lambert Publishing Company for taking the initiative in publishing this monograph.

Dibrugarh Tazid Ali
01/11/2010

CONTENTS

1 Introduction

So far as laws of mathematics refer to reality, they are not certain,
and as far as they are certain, they do not refer to reality

---Albert Einstein

1.1 Uncertainty and Fuzziness

In science as well as in mathematics, there are two views regarding uncertainty – the traditional view and the modern view. The traditional view is that uncertainty is undesirable and should be avoided as far as possible. The modern view is that uncertainty is a fact and it should not be ignored. It is generally agreed that an important point in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Lotfi A. Zadeh(1965), even though some ideas presented in the paper were envisioned some 30 years earlier by the Americal philosopher Max Black (1937). In this paper, Zadeh introduced a theory whose object - fuzzy sets – are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree. The significance of Zadeh's paper was that it challenged Aristotelian two-valued logic. When A is a fuzzy subset and x is a relevant object, the proposition "x is a member of A" is not necessarily either true or false, as required by two-valued logic, but it may be true only to some degree, the degree to which x is actually a member of A. The capability of fuzzy sets to express gradual transitions from membership to non-membership and vice versa has a broad utility. On one hand it provides a meaningful and powerful representation of measurement uncertainties, and on the other hand it provides a meaningful representation of vague concepts expressed in natural language.

A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is similar or compatible with the concept represented by the fuzzy set.

Research on the theory of fuzzy sets has been growing steadily since the inception of the theory in the mid-1960s. Zadeh's seminal paper has opened up new insights and applications in a wide range of scientific fields. Since then, the there has been an explosion of publications on fuzzy mathematics. Also a considerable body of literature has blossomed around the concept of fuzzy sets in wide range of areas.

Probability theory has been an age old and effective tool for modeling one type of uncertainty, which is characterized by randomness, i.e., processes in which occurrence of events is determined by chance. Fuzzy set deals will a different kind of uncertainty, uncertainty that may occur due to partial information, or due to unreliable information or due to inherent imprecision in the language with which the problem is defined.

Fuzzy set theory offers us a new angle to observe and investigate the relation between sets and their elements other than traditional *yes* or *no*'. The importance of fuzzy variables lies in the fact that they facilitate gradual transitions between states and, consequently, possess a natural capability to express and deal with observation and measurement uncertainties. Crisp variables do not have this flexibility. Since fuzzy variable models measurement uncertainties as part of experimental data, they are more close to reality than crisp variables. It is interesting to note that data based on fuzzy variables provide us with more accurate evidence about real phenomena than data based on crisp variables. Mathematics based on fuzzy sets has far greater expressive power than classical mathematics based on crisp sets. However usefulness of such representation depends on our capability to construct appropriate membership functions for various given concepts in various contexts.

Fuzzy set theory extended the basic mathematical concept – *set*. Since set theory is the cornerstone of modern mathematics, a new and more general framework of mathematics was established. Since then fuzzification started in divergent fields- both theoretical and applications. Algebra is one such branch in pure mathematics. Fuzzy algebra contains wider content than classical algebra. So it deepens the understanding of basic structure of classical algebra and offers new methods and results.

For sake of completeness the fundamentals of fuzzy set theory is dealt in length in this chapter. This I hope will help the beginners to understand the basic concepts so that they can venture the remaining chapters with confidence. Many examples are also presented in the chapter to support the results obtained. Wherever required fundamentals and preliminaries of the topic discussed is given in the chapter concerned.

1.2 Basic Notions of Fuzzy Set Theory

This section provides basic definitions of fuzzy set theory and its basic connectives.

1.2.1 Representation of Fuzzy set

L. A. Zadeh [51] introduced in his famous paper the following definition :

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership function which assigns to each object a grade of membership ranging from zero to one . So , a fuzzy set μ in a referential X is characterized by a membership function μ which associates with each x in X a real number in the interval [0 , 1] . The value of the membership function at an element $x \in X$ represents the "grade of membership" of x in μ . A fuzzy set μ of X is thus defined as a mapping

$$\mu : X \rightarrow [0, 1],$$

and it is a kind of generalization of the traditional characteristic function of a *subset A* of *X*

$$A : X \rightarrow \{0, 1\}.$$

Fuzzy sets are actually fuzzy subsets of X , as emphasized by Kaufmann [26] . This is why in the following we shall denote the membership grade of x to a fuzzy set μ as $\mu(x)$. The nearer the value of $\mu(x)$ to unity , the higher the grade of membership of x in μ . In particular , $\mu(x) = 1$ reflects full membership of x in μ , while $\mu(x) = 0$ express absolute non-membership of x in μ . Usual sets can be viewed as special cases of fuzzy sets where only full membership and absolute non-membership are allowed . They are called crisp sets . When $0 < \mu(x) < 1$, we speak of partial membership . A fuzzy set can also be denoted as a set of ordered pairs made of an element x in X and its membership grade :

$\{ (x , \mu(x)) : x \in X \}$. Henceforth we shall write fuzzy subset instead of fuzzy set .

Level Cuts . The α- cut μ_α of a fuzzy subset μ of X is the set

$\{ x \in X : \mu(x) \geq \alpha \}$, for $\alpha \in (0 , 1]$. The idea is to fix a positive α and to consider as members of the set the elements with membership grades equal or above α .

The set $\{ x \in X : \mu(x) > \alpha \}$, for $\alpha \in (0 , 1]$is called the strong α-cut and denoted by $\mu_{\alpha+}$.

Support . The support of a fuzzy subset μ of X is the set $\{ x \in X : \mu(x) > 0 \}$. It is usually denoted by μ^*.

Height. $\text{Sup}\{\mu(x) : x \in X\}$ is called height of μ. If height of a fuzzy set is 1 it is called a normal fuzzy set.

The fuzzy sets we are discussing here are called ordinary fuzzy sets. There are many generalizations of fuzzy sets. For example we have type 2-fuzzy set where the membership functions is itself a fuzzy set. In this sense ordinary fuzzy sets are also called type 1- fuzzy sets. Similarly type m-fuzzy sets are defined. Another generalization is given by Goguen[29], where membership function has its range in a lattice or a partially ordered set. Such fuzzy sets are called L-fuzzy sets. We will consider only ordinary fuzzy sets in all chapters except chapter 4 where L-fuzzy sets will be discussed.

1.2.2 Extension Principle

The following extension of set-theoretic functions to fuzzy subsets was proposed by Zadeh [51] .

Let f be a mapping from a set X into a set Y , and let μ be a fuzzy subset of X and ν be a fuzzy subset of Y . The fuzzy subset $f(\mu)$ of Y and the fuzzy subset $f^{-1}(\nu)$ of X , defined by

$\forall\ y \in Y$,

$f(\mu)(y) = \sup\{ \mu(x) : x \in X\ ,\ f(x) = y \}$ if $f^{-1}(y) \neq \phi$,

$= 0$, otherwise ,

and $\forall\ x \in X$,

$f^{-1}(\nu)(x) = \nu(f(x))$,

are called , respectively , the image of μ under f and the pre-image (or inverse image) of ν under f.

1.2.3 Basic Connectives

Zadeh [51] proposed extensions of inclusion , equality , complementation , union and intersection of classical sets to fuzzy subsets .

Inclusions . The inclusion of fuzzy subsets μ and ν of a set X is defined as follows :

$\mu \subseteq \nu \Leftrightarrow \forall\ x \in X\ ,\ \mu(x) \leq \nu(x)$.

The definition of inclusion leads to define equality between fuzzy subsets as follows :

$\mu = \nu \Leftrightarrow \mu \subseteq \nu$ and $\nu \subseteq \mu \Leftrightarrow \forall\ x \in X, \mu(x) \leq \nu(x)$ and $\nu(x) \leq \mu(x) \Leftrightarrow \forall\ x \in X,\ \mu(x) = \nu(x)$

.

For *fuzzy subsets* μ and ν of a *set* X and for $\alpha , \beta \in (0 , 1]$ the following results hold .

$\mu \subseteq \nu \Rightarrow \mu_\alpha \subseteq \nu_\alpha$, $\alpha \leq \beta \Rightarrow \mu_\beta \subseteq \mu_\alpha$, and $\mu = \nu \Leftrightarrow \mu_\alpha = \nu_\alpha$.

Intersection and Union . Other set-theoretic notions have been extended to fuzzy sets . As proposed by Zadeh [51] , the intersection and union of two fuzzy subsets μ and ν are given below :

$\forall\, x \in X$,

$(\mu \cap \nu)(x) = \min(\mu(x)\,,\, \nu(x)\,)$,

$(\mu \cup \nu)(x) = \max(\,\mu(x)\,,\, \nu(x)\,)$.

Complementation . Fuzzy subset complementation is defined by $\mu^c(x) = 1 - \mu(x)\ \ \forall\, x \in X$. It expresses the obvious requirement that the more x belongs to μ the less it belongs to its complement μ^c.

Fuzzy union, intersection and complement defined are called standard union, standard intersection and standard complement. There are many other definitions of union, intersection and complement. Some of the alternate definitions of fuzzy union (intersection, complement) are given below.

Union : Algebraic sum : $(\mu \cup \nu)(x) = \mu(x) + \nu(x) - \mu(x)\nu(x)$.

Bounded sum : $(\mu \cup \nu)(x) = \min\{\,1,\ \mu(x) + \nu(x)\}$

Drastic union: $(\mu \cup \nu)(x) = \mu(x)$, when $\nu(x) = 0$

$= \nu(x)$, when $\mu(x) = 0$

$= 1$, otherwise

Intersection : Algebraic product : $(\mu \cap \nu)(x) = \mu(x)\nu(x)$

Bounded difference : $(\mu \cap \nu)(x) = \max\{0, \mu(x) + \nu(x) - 1\}$

Drastic intersection : $(\mu \cap \nu)(x) = \mu(x)$, when $\nu(x) = 1$

$= \nu(x)$, when $\mu(x) = 1$

$= 1$, otherwise

Complement : (i) $\mu^c(x) = 1$, for $\mu(x) \leq t$

$= 0$, for $\mu(x) \geq t$

where $t \in [0, 1)$ is some threshold value which is context dependent.

(ii) $\mu^c(x) = \frac{1}{2}(1 + \cos\pi\mu(x))$

(iii) $\mu^c(x) = \{1 - \mu(x)\}/\{1 + \lambda\mu(x)\}$, where $\lambda \in (-1, \infty)$ is a parameter.

The selection of one definition or the other of fuzzy union (intersection, complement) will depend on the situation that is being depicted by the fuzzy sets. However we will stick to standard union (intersection, complement) throughout the discussion in this monograph.

1.2.4 Structural Properties

Naïve set theory is based on specific algebraic structure of subsets equipped with intersection , union and complementation : Boolean algebra . It would be good if the family of fuzzy subsets of a set X could be a Boolean algebra under suitable definitions of fuzzy connectives . However this is impossible mathematically : gradual membership and compositionality of membership grades are incompatible with Boolean structure . As a consequence some properties of Boolean algebras must be deleted for fuzzy subsets .

In the case of an ordinary subset F of a set X we have : $F \cap F^c = \phi$ and $F \cup F^c = X$.

But for a fuzzy subset μ of X it is not necessary that $\mu \cap \mu^c = 0$ and $\mu \cup \mu^c = 1$.

Choosing the min-max system with the complementation $1 - \mu(x)$ is the only way of preserving all properties of the Boolean structure , but the above two laws :

Associativity : $(\mu \cap \nu) \cap \sigma = \mu \cap (\nu \cap \sigma)$; $(\mu \cup \nu) \cup \sigma = \mu \cup (\nu \cup \sigma)$;

Commutativity : $\mu \cap \sigma = \sigma \cap \mu$; $\mu \cup \sigma = \sigma \cup \mu$;

Identity : $\mu \cap X = \mu$; $\mu \cup \phi = \mu$;

Absorption : $\mu \cap \phi = \phi$; $\mu \cup X = X$;

Idempotence : $\mu \cap \mu = \mu$; $\mu \cup \mu = \mu$;

De Morgan Laws : $(\mu \cap \sigma)^c = \mu^c \cup \sigma^c$; $(\mu \cup \sigma)^c = \mu^c \cap \sigma^c$;

Mutual distributivity : $\mu \cap (\sigma \cup \nu) = (\mu \cap \sigma) \cup (\mu \cap \nu)$; $\mu \cup (\sigma \cap \nu) = (\mu \cup \sigma) \cap (\mu \cup \nu)$;

Involution : $(\mu^c)^c = \mu$.

We note that the fuzzy subset inclusions are recovered from the connectives as follows : $\mu \subseteq \nu \Leftrightarrow \mu \cup \nu = \nu \Leftrightarrow \mu \cap \nu = \mu$.

1.2.5 Fuzzy Algebraic Structures

If the set X is equipped with some binary operation $*$ then a fuzzy subset μ of X will be closed under $*$ if its level-cuts are closed under $*$. In terms of the membership function, it comes down to define the closure of μ with respect to $*$ as :

$\forall\, x , y \in X$, $\mu(x*y) \geq \min(\mu(x), \mu(y))$.

For example a fuzzy subset μ of a groupoid G is called a fuzzy subgroupoid of G if $\forall\, x , y \in X$, $\mu(x*y) \geq \min(\mu(x), \mu(y))$.

The first examples of such a fuzzy extension of algebraic concepts are the fuzzy groups of Rosenfeld [46]. He showed how some basic notions of group theory should be extended in an elementary manner to develop the theory of fuzzy groups. He is the father of fuzzy abstract algebra. Since then, a considerable literature on fuzzy algebraic structures has been published. Liu in his pioneering paper [36] introduced and studied the concepts of fuzzy subrings and fuzzy ideals in rings. Fuzzy vector spaces over the field of real or complex field were discussed in Katsaras and Liu [31]. The concept of fuzzy field and fuzzy vector spaces over fuzzy fields were introduced and discussed by Nanda [45]. Nanda [46] proposed the concept of fuzzy lattice using the notion of fuzzy partial ordering.

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