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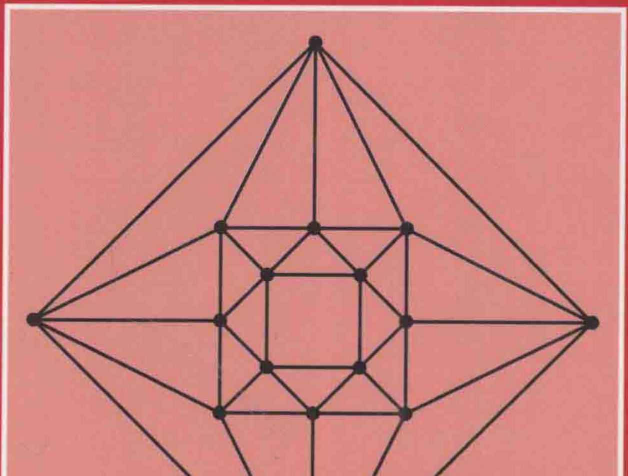
TOPICS IN STRUCTURAL GRAPH THEORY

Edited by
Lowell W. Beineke and Robin J. Wilson



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Topics in Structural Graph Theory

The rapidly expanding area of structural graph theory uses ideas of connectivity to explore various aspects of graph theory, and vice versa. It has links with other areas of mathematics, such as design theory, and is increasingly used in such areas as computer networks where connectivity algorithms are an important feature.

Although other books cover parts of this material, none has a similarly wide scope. Ortrud R. Oellermann (Winnipeg), internationally recognized for her substantial contributions to structural graph theory, acted as academic consultant for this volume, helping to shape its coverage of key topics. The result is a collection of 13 expository chapters, each written by acknowledged experts. These contributions have been carefully edited to enhance readability and to standardize the chapter structure, terminology and notation throughout. An introductory chapter details the background material in graph theory and network flows, and each chapter concludes with an extensive list of references.

LOWELL W. BEINEKE is Schrey Professor of Mathematics at Indiana University–Purdue University Fort Wayne, where he has been since receiving his Ph.D. from the University of Michigan under the guidance of Frank Harary. His graph theory interests are broad, and include topological graph theory, line graphs, tournaments, decompositions and vulnerability. With Robin Wilson he edited *Selected Topics in Graph Theory* (three volumes), *Applications of Graph Theory*, *Graph Connections*, *Topics in Algebraic Graph Theory* and *Topics in Topological Graph Theory*. Until recently he was editor of the *College Mathematics Journal*.

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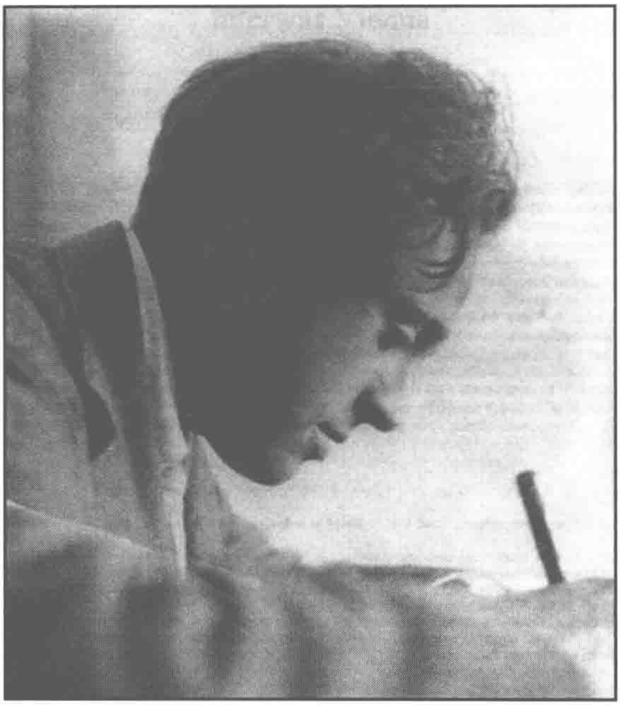

Karl Menger (1902–1985),
the founder of structural graph theory.

Foreword

Ortrud R. Oellermann

The overriding theme of this volume is connectedness in graphs. In its simplest form a graph is connected if every two vertices are connected by some path. Karl Menger's celebrated theorem changed the way we think about connectedness in graphs. The best-known version of Menger's theorem states that the maximum number of internally disjoint paths between a given pair of non-adjacent vertices in a graph equals the minimum number of vertices that separate the pair.

The connectivity of a graph is the minimum number of vertices whose deletion disconnects the graph. For a given integer k, a graph is k-connected if its connectivity is at least k. Menger's theorem can be used to establish Whitney's characterization of k-connected graphs: 'a graph is k-connected if and only if any two vertices are connected by at least k internally disjoint paths'. Another well-known result that follows from Menger's theorem is Dirac's cycle theorem: 'in a k-connected graph every set of k vertices lie on a common cycle'. Connectivity in graphs has given rise to a substantial body of work on minimally and critically k-connected graphs which is largely due to W. Mader.

An alternative formulation of Menger's theorem states: 'for given sets V and W of vertices in a graph G and a given integer k, there are k disjoint V–W paths in G if and only if every V–W separating set contains at least k vertices'; this is true in particular if V and W are disjoint sets of k vertices. So if V and W cannot be separated by fewer than k vertices, then there exist k disjoint paths, where each path has one end in V and the other end in W. If, for all such choices of V and W, one is able to specify the ends for each path in the collection of disjoint paths, then the graph is said to be *k-linked*.

W. T. Tutte's wheel theorem states: 'every 3-connected graph can be constructed from a wheel graph by repeatedly either splitting a vertex or by adding an edge between a pair of non-adjacent vertices'. Equivalently, 'every 3-connected graph, other than a wheel, can be reduced to a smaller 3-connected graph by either deleting or contracting an edge'. Thus every 3-connected graph has a wheel as a minor. As a result of Wagner's famous conjecture, the theory of graph minors was developed by

Neil Robertson and Paul Seymour who settled this conjecture in a series of papers. Since then, graph minors have played a fundamental role in many areas of graph theory, connectivity being no exception.

The theory of random graphs began in the 1960s, in a series of papers by P. Erdős and A. Rényi. One of their best-known and influential results in this area deals with the phase transition of the component structure in a typical random graph as the number of edges grows from less than half of the number of vertices to more than half.

Menger's theorem, Dirac's cycle theorem, Tutte's wheel theorem, graph minors, and Erdős and Renyi's work on phase transitions underpin many of the chapters in this volume. Menger's theorem is the basis for Chapter 1. Graphs whose connectivity equals the maximum degree are the subject of Chapter 2. Minimally and critically k-connected graphs and reduction methods for 3-connected graphs, first introduced by Tutte, serve as a platform for the material of Chapter 3, and contractible edges in k-connected graphs are further explored in Chapter 4. Dirac's cycle theorem serves as motivation for Chapter 5. The stronger version of the alternative formulation of Menger's theorem (k-linked graphs) and the work on graph minors play a fundamental role in Chapters 6 and 7. Measures of connectedness other than the connectivity are explored in Chapters 1, 7, 8 and 9. The work of Erdős and Renyi on random graphs inspired the results presented in Chapter 10. Network reliability depends on a probabilistic approach to connectedness in graphs and is the subject of Chapter 11. In Chapter 12 the evolution of deterministic algorithms for finding the connectivity of a graph are surveyed. The final chapter describes how different structures of graphs play a fundamental role in finding the best block designs.

Preface

The field of graph theory has undergone tremendous growth during the past century. As recently as the 1950s, the graph theory community had few members and most were in Europe and North America; today there are hundreds of graph theorists and they span the globe. By the mid-1970s, the subject had reached the point where we perceived a need for a collection of surveys of various areas of graph theory: the result was our three-volume series *Selected Topics in Graph Theory*, comprising articles written by distinguished experts and then edited into a common style. Since then, the transformation of the subject has continued, with individual branches (such as graph connectivity) expanding to the point of having important subdivisions themselves. This inspired us to conceive of a new series of books, each a collection of articles within a particular topics of graph theory written by experts within that area. The first two of these books were the companion volumes to the present one, on algebraic graph theory and on topological graph theory. This is thus the third volume in the series.

A special feature of these books is the engagement of academic consultants (here, Ortrud R. Oellermann) to advise us on topics to be included and authors to be invited. We believe that this has been successful, with the result being that the chapters of each book cover the full range of topics within the given area. In the present case, the area is connectivity, also called structural graph theory, with chapters written by authors from around the world. Another important feature is that, to the extent possible, we have imposed uniform terminology and notation throughout, in the belief that this will aid readers in going from one chapter to another. For a similar reason, we have not tried to remove a small amount of material common to some of the chapters.

We hope that these features will facilitate usage of the book in advanced courses and seminars. We sincerely thank the authors for cooperating in these efforts, even though it sometimes required their abandoning some of their favourite conventions – for example, computer scientists commonly use the term *node*, whereas graph theorists use *vertex*; not surprisingly, the graph theorists prevailed on this one. We also asked our contributors to endure the ordeal of having their early versions subjected to detailed critical reading. We believe that as a result the final product is thereby significantly better than it would otherwise have been (as a collection of individual chapters with differing styles and terminology). We want to express our heartfelt appreciation to all of our contributors for their cooperation in these endeavours.

We extend special thanks to Ortrud Oellermann for her service as Academic Consultant – her advice has been invaluable. We are also grateful to Cambridge University Press for publishing these volumes; in particular, we thank Roger Astley for his advice, support, patience and cooperation. Finally we extend our appreciation to several universities for the ways in which they have assisted with our project: the first editor (LWB) is grateful to his home institution of Indiana University–Purdue University Fort Wayne and also to Purdue University for an award of sabbatical leave during which he was a guest of the Mathematical Institute at Oxford University, while the second editor (RJW) has had the cooperation of the Open University as well as Keble College and Pembroke College, Oxford.

LOWELL W. BEINEKE
ROBIN J. WILSON

Contents

Preliminaries

LOWELL W. BEINEKE and ROBIN J. WILSON

1. Graph theory

This section presents the basic definitions, terminology and notation of graph theory, along with some fundamental results. Further information can be found in the many standard books on the subject – for example, Bondy and Murty [1], Chartrand, Lesniak and Zhang [2], Gross and Yellen [3] or West [5], or, for a simpler treatment, Marcus [4] or Wilson [6].

Graphs

A *graph* G is a pair of sets (V, E), where V is a finite non-empty set of elements called *vertices*, and E is a finite set of elements called *edges*, each of which has two associated vertices. The sets V and E are the *vertex-set* and *edge-set* of G, and are sometimes denoted by $V(G)$ and $E(G)$. The number of vertices in G is called the *order* of G and is usually denoted by n (but sometimes by $|G|$); the number of edges is denoted by m. A graph with only one vertex is called *trivial*.

An edge whose vertices coincide is a *loop*, and if two edges have the same pair of associated vertices, they are called *multiple edges*. In this book, unless otherwise specified, graphs are assumed to have neither loops nor multiple edges; that is, they are taken to be *simple*. Hence, an edge e can be considered as its associated pair of vertices, $e = \{v, w\}$, usually shortened to vw. An example of a graph of order 5 is shown in Fig. 1(a).

The *complement* $\overline{G}$ of a graph G has the same vertices as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Fig. 1(b) shows the complement of the graph in Fig. 1(a).

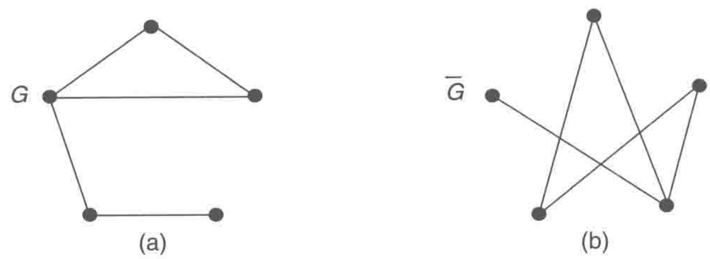

Fig. 1.

Adjacency and degrees

The vertices of an edge are its *endpoints* and the edge is said to *join* these vertices. An endpoint of an edge and the edge are *incident* with each other. Two vertices that are joined by an edge are called *neighbours* and are said to be *adjacent*; if v and w are adjacent vertices we sometimes write $v \sim w$, and if they are not adjacent we write $v \nsim w$. Two edges are *adjacent* if they have a vertex in common.

The set $N(v)$ of neighbours of a vertex v is called its *neighbourhood*. If $X \subseteq V$, then $N(X)$ denotes the set of vertices that are adjacent to some vertex of X.

The *degree* $\deg v$, or $d(v)$, of a vertex v is the number of its neighbours; in a non-simple graph, it is the number of occurrences of the vertex as an endpoint of an edge, with loops counted twice. A vertex of degree 0 is an *isolated vertex* and one of degree 1 is an *end-vertex* or *leaf*. A graph is *regular* if all of its vertices have the same degree, and is *k-regular* if that degree is k; a 3-regular graph is sometimes called *cubic*. The maximum degree in a graph G is denoted by $\Delta(G)$, or just Δ, and the minimum degree by $\delta(G)$ or δ.

Isomorphisms and automorphisms

An *isomorphism* between two graphs G and H is a bijection between their vertex-sets that preserves both adjacency and non-adjacency. The graphs G and H are *isomorphic*, written $G \cong H$, if there exists an isomorphism between them.

An *automorphism* of a graph G is an isomorphism of G with itself. The set of all automorphisms of a graph G forms a group, called the *automorphism group* of G and denoted by $\mathrm{Aut}(G)$. A graph G is *vertex-transitive* if, for any vertices v and w, there is an automorphism taking v to w. It is *edge-transitive* if, for any edges e and f, there is an automorphism taking the vertices of e to those of f. It is *arc-transitive* if, given two ordered pairs of adjacent vertices (v, w) and (v', w'), there is an automorphism taking v to v' and w to w'. This is stronger than edge-transitivity, since it implies that for each edge there is an automorphism that interchanges its vertices.

Walks, paths and cycles

A *walk* in a graph is a sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_k, v_k$, in which the edge e_i joins the vertices v_{i-1} and v_i. This walk is said to *go from* v_0 *to* v_k