# The Foundations of Geometry and the Non-Euclidean Plane

by George E. Martin

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# preface

This book is a text for junior, senior, or first-year graduate courses traditionally titled Foundations of Geometry and/or Non-Euclidean Geometry. The first 29 chapters are for a semester or year course on the foundations of geometry. The remaining chapters may then be used for either a regular course or independent study courses. Another possibility, which is also especially suited for in-service teachers of high school geometry, is to survey the the fundamentals of absolute geometry (Chapters 1–20) very quickly and begin earnest study with the theory of parallels and isometries (Chapters 21–30). The text is self-contained, except that the elementary calculus is assumed for some parts of the material on advanced hyperbolic geometry (Chapters 31–34). There are over 650 exercises, 30 of which are 10-part true-or-false questions.

A rigorous ruler-and-protractor axiomatic development of the Euclidean and hyperbolic planes, including the classification of the isometries of these planes, is balanced by the discussion about this development. Models, such as Taxicab Geometry, are used extensively to illustrate theory. Historical aspects and alternatives to the selected axioms are prominent. The classical axiom systems of Euclid and Hilbert are discussed, as are axiom systems for three- and four-dimensional absolute geometry and Pieri's system based on rigid motions.

The text is divided into three parts. The *Introduction* (Chapters 1–4) is to be read as quickly as possible and then used for reference if necessary. The formal axiomatic development begins in Chapter 6 of Part One, *Absolute Geometry* (Chapters 5–25). Chapter 5 contains a list of 15 models that are used throughout Part

One in discussing the relative consistency and independence of the axioms used in building our system. Isometries are introduced as soon as they are useful. In fact, the existence of the reflections is shown to be equivalent to the familiar SAS axiom. Chapter 25 shows that our five axioms for absolute geometry together with one of the equivalents of Euclid's Parallel Postulate (Theorem 23.7 gives 26 such equivalents) form a categorical system. Section 25.1 contains a detailed survey of the contents of Part Two, *Non-Euclidean Geometry* (Chapters 26–34). Although Part Two concentrates on hyperbolic geometry, many of the results have direct application to Euclidean geometry as well.

The classification of the isometries of the hyperbolic plane and, as a corollary, the classification of the isometries of the Euclidean plane appear in Chapter 29 of Part Two. In order to be sure of covering this important material in a one-semester or a two-quarter course it is suggested that Chapter 20 be finished halfway through the course. Chapters 10, 11, 15, and even 25 might be assigned as outside reading, postponed, or omitted. On the other hand, Chapter 30 should be included in such a course if time allows. (For a semester course meeting three times a week, the author uses the following schedule where exam days and reading days are omitted: 1–3, 4, 5, 6, 7, 8, 9, 9, 12, 13, 14, 16, 16, 17, 18, 19, 19, 20, 21, 21, 22, 22, 23, 23, 23, 24, 24, 26, 26, 27, 27, 28, 28, 28, 29, 29, 29.)

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# foreword to the student

"Thales, well known for his control of oil through a monopoly on the olive presses, today announced the invention of a means for obtaining knowledge. He calls the process *deduction.*" So began the front page story of the *Miletus Times* dated July 3, 576 B.C. An accompaning article reported the reactions of Oracle Joe to the invention. The utterances of Oracle Joe were deemed mysterious, as usual, and were quoted verbatim as follows: "Lines. O.J. sees parallel lines. Some seem more parallel than others in the hyperbolic plane. That's Non-Euclidean geometry. Just last week O.J. predicted that in a couple hundred years in a city near Egypt a guy named Euclid would make a big deal about parallel lines in a book that will endure as long as the stories of Homer. Euclid will use deduction. This deduction thing will hurt the oracle business, but the advice of oracles will be sought even into the Age of Aquarius. O.J. now sees tables, chairs, and beer mugs. Yes, it will be well over two thousand years and in worlds yet to be discovered before the implications and limitations of deduction begin to be fully realized. Non-Euclidean geometry will play an important role in all this. O.J. is never wrong—and is now open on Saturnday." With that we end the fantasy in this book but not, perhaps, the fantastic. (We shall see rectangles relegated to the domain of unicorns and pentagons with five right angles.)

There are many ways to distinguish between Euclidean and non-Euclidean geometry. The business about parallel lines is only one of the interrelated aspects whose totality is called *the theory of parallels.* To understand the theory of parallels we must begin our geometry almost from scratch. Thus we shall avoid the various traps that have ensnared mathematicians of the greatest genius. Also, the dynamics of building an axiom system very similar to but, in the end, vastly different from Euclid's are as exciting as any mystery novel. The story behind non-Euclidean geometry is one of the fascinating chapters in man's search for knowledge. In this text you will learn something of this story as well as the mathematical theory itself. For an appreciation of either, some understanding of the other is required. For those of you who may become teachers and feel non-Euclidean geometry is irrelevant, we quote the geometer Felix Klein: "After all, it is in order for the teacher to know a little more than the average pupil."

The following method is suggested for a quick, rough self-evaluation of your mastery of a particular chapter. After you have studied a chapter, answer each part of the True-or-False exercise in turn without allowing yourself to look ahead or to change an answer. Then score yourself, using the *Hints and Answers* section in the back of the book. If you missed a question because you forgot a definition from the theory, the *Index* will help you find the definition.

The author hopes that you enjoy your study of the theory of parallels.

# contents

## INTRODUCTION

The Introduction contains the prerequisites to our study of the foundations of geometry. In order to begin Part One, it is sufficient that the following questions be understood and answered: What is an equivalence relation on a set? What is a one-to-one mapping from one set onto another? What does it mean to say that an axiom system is consistent, independent, or categorical? The Introduction answers these specific questions and contains enough additional material so that almost every reader will encounter something new. It is recommended that these first four chapters be read as quickly as possible and then used for reference later if necessary.

# chapter one

## Equivalence Relations

### 1.1 LOGIC

We agree that a statement is either *true* or *false* (Law of the Excluded Middle) but not both (Law of Noncontradiction). Our use of "not," "and," "or," "if . . . then . . . ," and "iff" in relation to arbitrary statements $p$ and $q$ is explained by the *truth tables* in Table 1.1, where "T" stands for true and "F" for false. In mathematics "or" is always used in the inclusive sense. The conditional $p \Rightarrow q$ may be read in any one of the following equivalent ways:

**1** If $p$ then $q$.
**2** $q$ if $p$.
**3** $p$ only if $q$.
**4** $q$ or not $p$.
**5** $p$ is a sufficient condition for $q$.
**6** $q$ is a necessary condition for $p$.

The sentence "$p$ implies $q$" means that the conditional "if $p$ then $q$" is true. To say "(if $p$ then $q$) and (if $q$ then $p$)," we merely say "$p$ if and only if $q$" and write "$p$ iff $q$" or "$p \Longleftrightarrow q$."

Related to the conditional "if $p$ then $q$" are its *converse* "if $q$ then $p$" and its *contrapositive* "if not $q$ then not $p$." It should be easy to think of a conditional which is true but whose converse is false. On the other hand, a conditional is true if and only if its contrapositive is

TABLE 1.1

| p | q | not p | p or q | p and q | if p then q | p iff q |
|---|---|---|---|---|---|---|
| T | T | F | T | T | T | T |
| T | F | F | T | F | F | F |
| F | T | T | T | F | T | F |
| F | F | T | F | F | T | T |

true. One way of convincing yourself of this is to observe that the following are all equivalent: (1) If not $q$, then not $p$. (2) (Not $p$) or not (not $q$). (3) (Not $p$) or $q$. (4) $q$ or not $p$. (5) If $p$, then $q$. Another way is to check the truth table in Table 1.2, where the numbers at the bottom indicate the order in which the columns were entered in constructing the table.

You intuitively know the meaning of the two quantifiers that are used in basic logic. One is the *existential quantifier,* which may be denoted by any one of the following: there exists, there exist, there is, there are, for some. The other is the *universal quantifier,* which may be denoted by any one of the following: for any, for all, each, every. Actually, the universal quantifier may be logically defined in terms of the existential quantifier and negation. For example, if $p$ denotes some proposition about the integers, then "for all integers, $p$" means the same thing as "there does not exist an integer such that not $p$." One thing to look out for is that the little words *a, an,* and *the* are often hidden quantifiers in English. For example, "The diameters of a circle intersect at a point" contains three quantifiers and means that *for any* circle *there exists* a point such that *each* diameter of that circle passes through that point.

Consider the statement "If $N$ is a positive integer, then $N^2 - 79N + 1601$ is a prime." To prove this statement it would not be sufficient to show that $N^2 - 79N + 1601$ is a prime for several values of $N$. Even to show that you get a prime for the first seventy-nine positive integers is not a proof of the statement. Actually, the statement is false as $N^2 - 79N + 1601 = 41^2$ when $N = 80$. Note that one case where the statement is false proves that the statement is false! In other words, it only takes one *counterexample* to disprove a statement.

TABLE 1.2

| p | q | (p ⇒ q) | iff | ((not q) | ⇒ | (not p)) |
|---|---|---|---|---|---|---|
| T | T | T | T | F | T | F |
| T | F | F | T | T | F | F |
| F | T | T | T | F | T | T |
| F | F | T | T | T | T | T |
| 1 | 2 | 3 | 7 | 4 | 6 | 5 |

## 1.2 SETS

Most of us have heard that a *set* is a collection of elements. "$x \in A$" means that $x$ is an element of set $A$; "$x \notin A$" means that $x$ is not an element of set $A$. The statement that set $A$ is a *subset* of set $B$ is written "$A \subset B$" and means $x \in A$ only if $x \in B$. The set of all positive integers is a subset of the set of all integers. Some sets can be exhibited explicitly. For example, the set of odd digits is $\{1, 3, 5, 7, 9\}$. Often it is impractical or impossible to list the elements of a set. If $\mathbf{R}$ is the set of all real numbers, we may denote the set of all positive reals by "$\{x|x \in \mathbf{R}, x > 0\}$" and read "the set of all elements $x$ such that $x$ is a real number and $x$ is greater than zero."

Let $A$ and $B$ be sets. The *union, intersection, difference,* and *Cartesian product* of $A$ and $B$ are defined, respectively:

$$A \cup B = \{x|x \in A \quad \text{or} \quad x \in B\},$$
$$A \cap B = \{x|x \in A \quad \text{and} \quad x \in B\},$$
$$A \setminus B = \{x|x \in A \quad \text{but} \quad x \notin B\},$$
$$A \times B = \{(x, y)|x \in A, \quad y \in B\}.$$

Since "but" means "and" in mathematical logic, we see that $A \setminus B$ is the set of all elements of $A$ that are not also elements of $B$. Note that $A \times B$ is just the set of all ordered pairs such that the first element is in $A$ and the second element is in $B$.

If $A$ and $B$ are set with no element in common, then $A$ and $B$ are *disjoint*. In this case we write "$A \cap B = \varnothing$." So $\varnothing$ is the set which contains no elements and is called the *empty set* or *null set*. The empty set is a subset of every set. Two sets *intersect* if they are not disjoint.

If $L$ and $R$ are sets, then $L = R$ iff $L \subset R$ and $R \subset L$. One may exercise his ability to use "and" and "or" by proving the following distributive laws, where $A, B, C$ are sets:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

We may wish to speak of a set of sets. In this case the elements of the set are subsets of some other set. For example, $\{\{1, 2, 3\}, \{3, 4, 5, 6\}\}$ is a set with exactly the two elements $\{1, 2, 3\}$ and $\{3, 4, 5, 6\}$. Note that for general element $S$, we have $S \neq \{S\}$. In particular, $\varnothing \neq \{\varnothing\}$