Introductory Mechanics

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Preface

It can no longer be assumed that all students of mechanics will already have met the subject before entering higher education. Undergraduate courses must therefore be designed to interest and challenge those students who have studied the subject before and yet to remain within the grasp of those studying the subject for the first time. The aim of this book is to present such a course of mechanics, introductory in nature but meeting the standards usually associated with a first year undergraduate course. The course is suitable for both mathematicians and physicists and also for school teachers wishing to gain further insight into the subject.

Two mathematical models of motion are introduced, namely Newton's classical mechanics and Einstein's relativistic mechanics. Emphasis is placed on the understanding of concepts; topics including inertial frames and the equivalence of inertial mass and gravitational mass being treated in detail. Such emphasis is not made at the expense of manipulative skills; indeed, many worked examples and exercises are included at the end of each chapter. The general theory is developed in three dimensions using vectors, but most of the applications are confined to two dimensions. SI units are used throughout and, whenever possible, the symbols used are those recommended by the Royal Society.

Continuum mechanics and mathematical physics have both developed from the study of the motion of particles. Chapters on fluid mechanics and on Newtonian cosmology are therefore included at the end of the book in order to illustrate these developments. Even if these chapters are not included in the first year undergraduate courses the author would hope that they might whet the appetite of the readers and encourage them to continue their studies into the more advanced aspects of mechanics.

I would like to acknowledge the contribution made by my wife Carole and to thank her for typing the manuscript and for her encouragement during the writing of this book.

Hull
1980

CDC

Glossary of symbols

mass	m
inertial mass, gravitational mass, rest mass	m_i, m_g, m_0
time	t
position vector	$\mathbf{r}$
centre of mass	C
position vector of centre of mass*	$\mathbf{c}$
frame of reference	$\mathbf{S}$
centre of mass frame†	$\bar{\mathbf{S}}$
laboratory frame	$\mathbf{S}_L$
velocity between two frames	$\mathbf{V}$
linear momentum	$\mathbf{p}$
angular momentum	$\mathbf{L}$
moment of inertia about a point O	I_O
moment of inertia about an axis l	I_l
force	$\mathbf{F}$
moment of force	$\mathbf{M}$
moment of couple	$\mathbf{G}$
work	W
total energy	E
potential energy	V, ϕ
kinetic energy	T
modulus of elasticity	λ
frequency	ν
angular frequency	ω
periodic time	T
damping coefficient	k
coefficient of friction	μ
viscosity	η
density	ρ
pressure	p
Reynolds number	R_e
electric charge	Q
electric field vector	$\mathbf{E}$
magnetic induction vector	$\mathbf{H}$

* The symbol $\mathbf{c}$ is also used occasionally for the connecting vector between two origins O and O′, i.e. $\mathbf{c} = \overrightarrow{OO'}$.
† A bar is frequently used to denote a quantity defined relative to the centre of mass, e.g. $\bar{p}$.

SI units

Quantity	Unit	Abbreviation
length	metre	m
mass	kilogram	kg
time	second	s
frequency	hertz	Hz (or s^{-1})
force	newton	N (or $kg\,m\,s^{-2}$)
energy	joule	J (or $kg\,m^2\,s^{-2}$)
power	watt	W (or $kg\,m^2\,s^{-3}$)
angle	radian	rad
electric charge	coulomb	C
electrostatic potential	volt	V
electric field		$V\,m^{-1}$
magnetic induction	tesla	T

All the equations which appear in the study of mechanics are dimensionally homogeneous, that is the terms of each equation all have the same SI units. If any one term in an equation has different units to the other terms then a mistake has been made – a test of accuracy which proves most useful with experience. The fact that the equations are dimensionally homogeneous leads to *dimensional analysis*, a method of solving problems which is widely used by engineers and research mathematicians.

Physical constants

Universal constants

gravitational constant, $G = (6.670 \pm 0.006) \times 10^{-11}\,\mathrm{m^3\,kg^{-1}s^{-2}}$
speed of light in a vacuum, $c = 2.997925 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Planck's constant, $h = 6.6256 \times 10^{-34}\,\mathrm{J\,s}$

Astronomical constants

(i) *the earth*
acceleration due to gravity (at Potsdam), $g = 9.81274\,\mathrm{m\,s^{-2}}$
mass $= 5.976 \times 10^{24}$ kg
equatorial radius $= 6.378160 \times 10^{6}$ m
polar radius $= 6.356775 \times 10^{6}$ m
average distance from the sun $= 1.496 \times 10^{11}$ m

(ii) *the moon*
mass $= 7.350 \times 10^{22}$ kg
equatorial radius $= 1.722 \times 10^{6}$ m

(iii) *the sun*
mass $= 1.990 \times 10^{24}$ kg
equatorial radius $= 6.960 \times 10^{8}$ m

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1

Mathematical modelling

1.1 General discussion

The pure mathematician is interested in studying mathematical structures. The applied mathematician uses these mathematical structures to describe observed phenomena. The description of observed phenomena by known mathematical structures is called mathematical modelling, and the mathematical structures are said to constitute a mathematical model of the observed phenomena. Mathematical modelling plays a central role in many disciplines, for example in physics, chemistry, the biological sciences, economics and environmental studies. The crucial difference between the Arts and the Sciences is that the scientist is interested in just those phenomena which can be modelled mathematically. The chemist who performs an experiment to find a reaction between substances will model his observations in terms of a symbolic equation. The geographer constructs mathematical models describing the growth of population in urban and rural communities. The biologist models birth and death processes by means of a differential equation.

Once a mathematical model of some observed phenomena has been formulated, then one can manipulate the mathematics and, hopefully, predict the occurrence of some new phenomena. It is this element of prediction which excites the interest of the academic and which can be of benefit to society. If the predictions are found to occur in reality then the mathematical model is satisfactory. However, once a prediction fails to agree with reality, then the mathematical model becomes incomplete and must be replaced by a more refined model. The whole process of setting up a mathematical model of observed phenomena and then testing the predictions against reality is often referred to as the 'Scientific Method' and is usually attributed to Descartes.

As an example of mathematical modelling, suppose that football supporters going to a match can park their cars in either of two cul de sacs, denoted hereafter by P_1 and P_2. The observed rates at which cars enter P_1 and P_2 can be modelled as two real numbers c_1 cars/second and c_2 cars/second (see Fig. 1.1). If at time $t = 0$ there are no parked cars, then one can predict that the number of cars parked in P_1 and P_2 at time t seconds will be $c_1 t$ and $c_2 t$ respectively (it is assumed here that the rates c_1 and c_2 are both constants). If $c_1 > c_2$ then the number of cars parked in P_1 will, at all times, be greater than the number of cars parked in P_2. One would predict from this that the first cul de sac will fill up before the second, and this

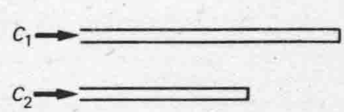

Figure 1.1

prediction might well be of use to the policeman controlling traffic to the football match. Unfortunately this prediction may not occur in reality because no account has been taken of the fact that once a cul de sac fills up then the number of cars in that cul de sac remains constant, i.e. the formula ct for the number of parked cars does not remain valid. The mathematical model needs to be modified by taking into account the total number of cars which can be parked in P_1 and P_2. Let these numbers be n_1 and n_2. The first cul de sac will fill up at a time t given by $c_1 t = n_1$, i.e. at time $t = n_1/c_1$. Similarly the second cul de sac will fill up at time $t = n_2/c_2$. The required prediction can now be made by determining which is the smaller of the two ratios, n_1/c_1 or n_2/c_2.

In the process of modelling it is often necessary to make approximations. Usually the phenomena being modelled are very complex and so, in order to achieve a reasonably simple mathematical model, it is necessary to neglect certain aspects of the phenomena and to model only those dominant aspects which, intuitively, are of importance. For example, in the parking problem already discussed, the number of cars which can be parked in each cul de sac is very difficult to estimate. These numbers depend upon the length and curvature of the cul de sacs, the length and breadth of each individual car, and finally the spaces left between parked cars. In order to achieve a simple mathematical model suppose that the cul de sacs are straight and of lengths l_1 and l_2, that the cars are each of length l (this can be taken to be the average length of all cars produced) and that the cars are parked bumper to bumper. Then the total number of cars which can be parked in P_1 and in P_2 is $n_1 = l_1/l$ and $n_2 = l_2/l$. This model is obviously very naive. One cul de sac might contain a high class restaurant which attracts some of the wealthier supporters for a pre-match lunch. Then the average length of the cars in that cul de sac will probably be greater than the average length of the cars in the other cul de sac and this difference must be taken into account. A difficulty arises here because one can never know *beforehand* whether, at any particular match, the restaurant will be used by the supporters and so influence the parking of cars. At best one can estimate the probability that the restaurant will be used, and then the mathematical model becomes a probabilistic model rather than a deterministic model.

It is possible that the applied mathematician has difficulty in finding a suitable mathematical structure with which to model some particular observed phenomena. This might lead to a search for new, as yet unformulated, mathematical structures. Such a search belongs to the realm of pure mathematics. In this way applied mathematics can often prompt and motivate the discovery of 'new' pure mathematics. In fact the distinction between 'pure mathematicians' and 'applied mathematicians' was, in the past, far less apparent than it is now. Such men as Newton, Euler and Hamilton are all famous both as pure and as applied mathematicians.

Occasionally the whole process of mathematical modelling becomes inverted so that the phenomena being modelled are able to yield information about the mathematical structure being used. For example, the behaviour of an electrical circuit can be modelled mathematically and it is found that the current flowing in the circuit satisfies a certain differential equation. If the circuit is very complex, then the differential equation is very complex and therefore difficult to solve. However, it is a trivial matter to measure the current flowing in the circuit and this measured value is of course the required solution to the differential equation. This method of 'solving' differential equations is the basis of the modern electronic analogue computer.

It may also happen that two phenomena from different disciplines are modelled by the same mathematical structure. The two phenomena are then analogues of each other. As an example of this situation, consider the economist's problem of constructing the cheapest road system between given towns. The given towns can be modelled as points on a plane and the roads can be modelled as straight line segments lying on the plane, as shown in Fig. 1.2. If the cost/kilometre of roadway is the same at all points, and if the cost of constructing junctions is neglected, then the cheapest road system will be that of minimal total length. The mathematical determination of such a road system is very difficult. Now consider two parallel plates of glass a distance d apart. Suppose a map of the region containing the towns is etched onto one plate of glass and suppose that perpendicular rods are placed between the plates of glass at the location of the towns. If this is dipped into a soapy solution, then bubbles will form between the rods, the bubbles being perpendicular to the plates of glass. Viewed from the plate of etched glass these bubbles will form a network of line segments connecting the towns. The effect of surface tension is to minimize the surface area of the bubbles so that the bubbles will move until their area, which is just the constant d times the total length of the network, is a minimum. Hence in its final configuration the network will be a network of minimal total length and will therefore correspond to the cheapest road system between the towns (see Fig. 1.3). This is an example of a physical analogue of a problem in economics.

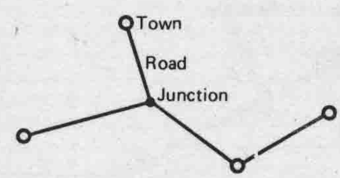

Figure 1.2

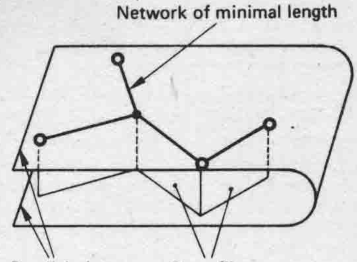

Figure 1.3

1.2 What is mechanics?

Having given a general discussion of mathematical modelling, it is now appropriate to explain just what is meant by *mechanics* and to give a short outline of the scope of this text. Mechanics is the study of mathematical models describing the relationship between the phenomena which are observed to cause bodies to move and the observed motion of those bodies. It is not to be confused with *kinematics* which is the study of mathematical models describing only the observed motion of the bodies. The important difference between kinematics and mechanics is that given some information about the initial motion of a body, mechanics alone will *predict* the subsequent motion of that body. Thus mechanics plays a far wider role in the physical sciences than does kinematics. It is often possible, with very little loss of accuracy, to model bodies of finite size as particles. This is certainly true, for example, when discussing the motion of a planet around the sun. It is sufficient therefore to begin our study of mechanics by modelling the motion of a particle rather than the motion of a body of finite size. This simple model can then be extended to the motion of bodies of finite size. The motion of such bodies depends upon the physical properties of the materials of which the bodies are composed. Here attention will be restricted to bodies composed of rigid materials and, to a far less extent, bodies composed of fluid materials. Of course elastic, thixotropic, plastic, and other materials are also of great interest but are beyond the scope of this book.

Two different types of mathematical models of motion will be described here. The first is the classical model due to Newton, the study of which is called *Newtonian mechanics* or *classical mechanics*. The second is the special relativistic model due to Einstein, the study of which is called *relativistic mechanics*. Although Einstein's model has superseded Newton's model, it is found that in most instances classical mechanics leads to predictions which agree well with observations. It is only when speeds approach the speed of light that classical mechanics is inadequate and has to be replaced by relativistic mechanics. For this reason, most attention will be paid here to classical mechanics.

Both models have in the past led, and still lead, to predictions of new and amazing phenomena. For example, application of classical mechanics to the orbit of Uranus enabled Adams and Leverrier to predict, in 1846, the presence of a hitherto unknown planet observed by Galle in 1847 and named Neptune. Application of relativistic mechanics enabled Einstein to predict, in 1905, the existence of nuclear energy almost forty years before the manufacture of the first atom bomb. Very many similar predictions could be quoted and the success of the two models must rank as one of the greatest achievements of mankind.

1.3 The classical models of time and space

Naturally occurring events have led man to the concept of time. Observation of the rhythm of the seasons led to an estimate of the length of the year as a number of days, the day being the obvious interval between two sunrises. The advent of agriculture necessitated a more accurate 'calibration' of the seasons, and this was achieved, for example, by observing the motion of the sun. The length of the year was specified by observing midsummer's day – the day on which the sun reaches its greatest height in the sky. The phases of the moon led to the definition of the month, and the rising and setting of the sun divided the day into two intervals (day and night). Further domestication of man led to the day being further subdivided, and this was done by observing the position of the sun (or equivalently the length of a shadow). Better astronomical observations made the subdivisions of the day smaller, and the regularity of the swinging of a pendulum, and similar devices, led to the construction of instruments. Using these instruments the interval of time between the occurrence of two events can be modelled by a real number, for example by counting the number of swings performed by a pendulum between the occurrence of the two events. Thus in the Newtonian (classical) model, intervals of time are modelled by real numbers and these real numbers are measured by instruments known as clocks. The real number which models a given interval of time will depend upon the unit of time used to calibrate the clocks. Here the unit of time used will be the second. The time at which some event E occurs is then simply the interval of time between the occurrence of some prescribed event which is used to define the origin of time and the occurrence of the event E. For example, the time of day is the interval of time between midnight and the present.

Different clocks (calibrated in seconds) often given different readings. This can be for one of two reasons. The first is that one of the clocks is mechanically faulty, in which case it must be replaced. The second is that the clocks are not synchronized, i.e. they are using different prescribed events to define the origin of time. The important assumption made in the classical model of time is that once two identical clocks are synchronized then they will remain synchronized even when the clocks are in relative motion. Such a model is said to assume the existence of a *universal time*. Few readers would question the existence of a universal time. For example, if two friends are arranging to meet, then they might well check initially that their wristwatches show the same time but they will subsequently assume without question that their wristwatches will remain synchronized, enabling them to arrive at their meeting place simultaneously.

The path of a moving particle is a curve in space. In order to model this curve, and the motion of the particle, it is necessary to have a mathematical model of space itself. The classical (Newtonian) model of space is simply a three-dimensional euclidean geometry. This mathematical model of physical space will be referred to as *euclidean space*. Notice that there is no suggestion here that physical space *is* euclidean, and it is important when referring to euclidean space to remember that this is simply the geometry which is chosen as the mathematical model of physical space. In fact physical space is not always modelled by a euclidean geometry. The first mathematician to formulate a non-euclidean geometry was Gauss. He did not publish his work because it ran contrary to the contemporary view which identified euclidean geometry with physical space. It was only in 1830, after Bolyai and Lobachevsky had independently published accounts of non-euclidean geometry, that Gauss announced his results which had been obtained thirty years earlier. Gauss had been motivated in his work on non-euclidean geometry by his interest in the study of the shape and dimensions of the earth's surface. A student of his, Riemann, developed Gauss's ideas into a general theory, and Einstein chose as the mathematical model of space and time for his general theory of relativity a four-dimensional Riemannian geometry. Thus in Einstein's general theory of relativity, which is a theory of gravitation, physical space is not modelled by a euclidean geometry. It is therefore fortunate that the classical model of space is a three-dimensional euclidean geometry, a mathematical structure with which readers will be familiar.

1.4 Euclidean geometry

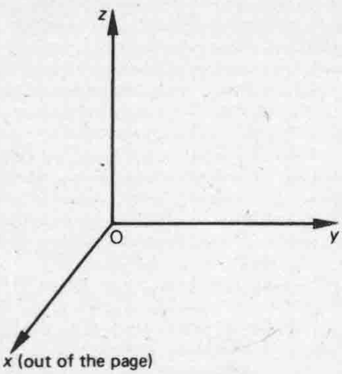

Figure 1.4

The position of a point P in a three-dimensional euclidean geometry can be specified by the coordinates (x, y, z) of the point relative to some given set of mutually orthogonal axes Ox, Oy, Oz. Such coordinates are called (rectangular) *cartesian coordinates*, being named after Descartes. The fact that the axes are mutually orthogonal does not specify their relative orientation completely. Here the axes will always be chosen so that their relative orientation is as illustrated in Fig. 1.4. Such axes are called *right handed.*

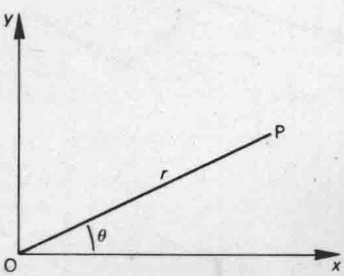

Figure 1.5

Although cartesian coordinates are very often used to specify the position of a point P they are not the only coordinate system available, nor are they always the most convenient. It is not necessary to give here a general discussion of alternative coordinate systems.* However, when specifying the position of a point P in a two-dimensional euclidean geometry, it is often convenient to use polar coordinates. These coordinates are defined by introducing a line l and an origin O lying on the line. The polar coordinates of the point P are then (r, θ) where r is the length of OP and θ is the angle between OP and the line l with $0 \leq \theta < 2\pi$. By definition, $r \geq 0$. If cartesian axes are chosen as in Fig. 1.5 then the polar coordinates (r, θ) and the cartesian coordinates (x, y) of the point P are related by the equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{1.1}$$

Notice that each point is specified by one and only one set of cartesian coordinates (x, y) and that the same is true of polar coordinates (r, θ) except for the origin O which is specified by $(0, \theta)$ for all values of θ.

The use of specific coordinate systems is to be discouraged when giving a general discussion of euclidean geometry or of mathematical models of motion based on a euclidean space. For this reason, general results will be formulated in terms of vectors, and specific coordinate systems will only be used when dealing with particular problems. Then the type of coordinate system will be chosen to make the solution of the problem as simple as possible. Thus the point P is specified by the vector $\overrightarrow{OP}$ which is called the *position vector of P relative to O.* The position vector is denoted by $\mathbf{r}$. In order to specify a vector $\mathbf{v}$ defined at the point P, it is necessary either to specify the magnitude and direction of the vector or to specify the components of the vector relative to a set of basis vectors defined at the point P. Associated with any given coordinate system there is, at each point P, a naturally defined set of basis vectors. Thus for two-dimensional cartesian coordinates (x, y) there is defined at each point P a set of basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ where $\hat{\mathbf{i}}$ is a unit vector parallel to Ox and $\hat{\mathbf{j}}$ is a unit

*See Chapters 5 and 12 of the author's book *Introductory Vector Analysis* published by Edward Arnold, 1974.

vector parallel to Oy (see Fig. 1.6). Notice that $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ form an orthonormal basis and that

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \tag{1.2}$$

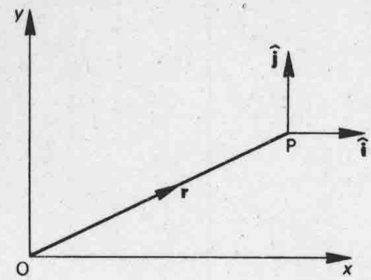

Figure 1.6

For polar coordinates (r, θ) there is defined at each point P a set of basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ where $\hat{\mathbf{r}}$ is a radial unit vector in the direction of $\overrightarrow{OP}$ and $\hat{\boldsymbol{\theta}}$ is a transverse unit vector perpendicular to $\overrightarrow{OP}$ and orientated in the sense of increasing θ (see Fig. 1.7). Again $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ form an orthonormal basis and

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{1.3}$$

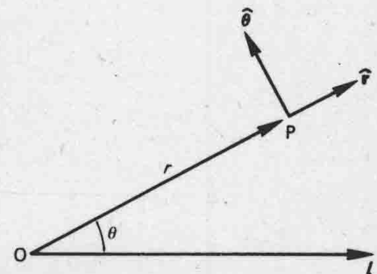

Figure 1.7

The fact that these two sets of basis vectors are orthonormal makes it very easy to write down the components of a given vector $\mathbf{v}$ relative to either set of basis vectors. In fact each component of $\mathbf{v}$ is found by multiplying the magnitude of $\mathbf{v}$ by the cosine of the angle between the vector $\mathbf{v}$ and the relevant basis vector. In particular, the vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ can be written in terms of the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ as

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} \\ \text{and}\quad \hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}} \end{aligned} \tag{1.4}$$

Inverting these equations algebraically yields

$$\begin{aligned} \hat{\mathbf{i}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \\ \text{and}\quad \hat{\mathbf{j}} &= \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}} \end{aligned} \tag{1.5}$$

The basis vectors associated with a three-dimensional cartesian coordinate system (x, y, z) are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. Such a basis is called a *cartesian basis* and the components (v_x, v_y, v_z) of a given vector $\mathbf{v}$ relative to a cartesian basis are called the *cartesian components* of the given vector. Thus

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \tag{1.6}$$

The equation (1.2) generalizes to

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{1.7}$$

and so the cartesian coordinates of any point are just the cartesian components of the position vector of that point.

1.5 Differentiation of vectors

The vectors met in mechanics usually describe physical properties of some moving particle. As the particle moves, the point P in euclidean space which models the location of the particle in physical space will describe a curve. The position vector of the point P lying on the curve will therefore be a function of some parameter λ, that is

$$\mathbf{r} = \mathbf{r}(\lambda) \tag{1.8}$$

Any vector $\mathbf{v}$ defined at P may now vary in magnitude and direction as the location of the point P varies. The vector $\mathbf{v}$ is then called a vector function (more correctly a vector-*valued* function) of λ and is written as

$$\mathbf{v} = \mathbf{v}(\lambda) \tag{1.9}$$