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Calculus
An Integrated Approach

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CALCULUS: An Integrated Approach

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CALCULUS

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To

Margaret Small

for her love, support, and unlimited patience

PREFACE

To the Instructor

This text presents an *integrated* approach to calculus by integrating the development of major concepts with respect to both single and multivariable functions. The emphasis is on developing a conceptual and unified approach. A heavy reliance on approximation and error bound analysis provides a unifying theme throughout the text. Chapters one through four treat the Differential Calculus and Chapters five through nine the Integral Calculus. It is expected that this material represents two semester's worth of material (i.e., 45 to 55 class periods per semester).

The intended audience for this text are well prepared students. In particular, students who have successfully completed a full year course in calculus in high school. The Mathematical Association of America (MAA) recommends that colleges design calculus courses for these students that:

1. acknowledge and build on the high school (calculus) experiences of the students,
2. provide necessary review opportunities to ensure an acceptable level of understanding of Calculus I topics,
3. are *clearly different* from high school calculus courses (in order that students do not feel that they are essentially just repeating their high school work),
4. result in an equivalent of a one semester advanced placement.

This text satisfactorily addresses each of these four recommendations. In particular, integrating the development of major concepts with respect to single and multivariable functions makes the text (and course) *clearly different* from high school courses.

The authors' primary concerns are with the development of the student's ability to learn, analyze, and solve problems; to learn how to learn mathematics; and to become involved in developing mathematics. These concerns are reflected in the following aspects that are emphasized throughout the text.

- The Basic Approximation Process (approximate–improve approximation–generate sequence of approximations–take limit). This process underlies the development of every major concept in the Calculus.

- The "natural" role of sequences in the study of convergence. The limit of a sequence is the basic limit idea. The limit of a function is obtained by first composing the function with an appropriate sequence and then evaluating the limit of the resulting composition sequence.

- Development of an inquisitive approach on the part of the reader. In addition to the homework exercises, there are three categories of questions included in the text that are designed to test understanding and provoke thought.

  - Questions that are posed and answered in the body of the text.
  - Questions for which no answers are given. These are often used to call the reader's attention to missing steps in a calculation.
  - Questions that are set off by themselves in the text for which detailed answers are given in the Appendix.

- The structure for the development of a mathematical concept. There are four major stages in the development process:

  - Motivational stage. Particular instances "rooted" in student experiences are used to establish a "desire" and a "need" to develop the particular concept in question.
  - Definition stage. This is the concluding step in the Basic Approximation Process.
  - Algebraic stage. Determining how the concept "behaves" with respect to the standard arithmetic and functional operations.
  - Application stage. Employing the concept in particular applications as well as considering the concept as part of a mathematical theory.

- Development of conceptual understanding rather than a compilation of facts and techniques.

- Non-routine exercises that require the student to approach material in different ways: True-False questions, Give an example or show why no example can exist, Prove or Disprove the given statement, Analysis by graphing, and Projects. In order to actively involve students in developing mathematics, the proofs of several important theorems have been converted into exercises (with ample hints). "Project" exercises are open-ended exercises that lend themselves to small group efforts. The authors suggest that these exercises culminate in written reports.

- The text contains the usual routine exercises, although we are not in the competition for the text with the largest number of exercises. The routine exercises consist of drill exercises on important manipulations and word problems in which the student must convert a verbal description into a mathematical problem. These problems provide the student with useful practice in interpretation as well as suggesting the wide range of possible applications.

The importance of questioning, conjecturing, and working numerical examples cannot be over emphasized. (Mathematics is not a spectator sport!) Surely an inquisitive approach and a willingness to "struggle" with a problem are marks of an educated person. Students are encouraged to work in small groups as well as individually, and to utilize both the computer and hand calculators for their computations.

Additional features that characterize this work are:

- Numerical methods (e.g., the bisection algorithm, numerical integration) are used in the constructive development of concepts rather than appended in optional sections.

- Exercises involving the use of several levels of computational resources. It is generally recognized that numerical methods are best carried out with the aid of a calculator or computer. Computer Algebra Systems (such as Maple, MACSYMA, muMATH, Mathematica, SMP, Derive, etc.) are becoming available and will become equally important for carrying out algorithmic processes involving symbolic manipulation such as differentiation. There are exercises throughout the text whose basic aim is to increase student involvement with mathematics through exploratory exercises involving a calculator, graphing program, or computer. Students gain little by passively watching a computer print out an answer; they can gain a great deal by actively using the computer to explore, conjecture, and verify.

- Careful development of the function concept (source, domain, range, target) and consistent use of the notation $f : \mathbb{R}^n \to \mathbb{R}^m$. Helping students learn how to analyze functions is a major goal of this text.

- Special emphasis is placed on interpreting graphs. It is expected that students will learn to use graphs to "guide their analysis" rather than following the traditional approach of using analysis to produce a graph. For example, extremal values are introduced by analyzing the "high" and "low" points of graphs. Other examples include studying the relationship between a function and its derivative by sketching the graph of one from the graph of the other. (What does the location of an extreme value of the derivative function imply about the graph of the function?)

- A conscious effort to guide students in understanding the logic involved in the statement of a theorem and its proof. An initial step in learning how to prove theorems is practicing making up examples. (Most exercise sections contains several such questions.) A "next" step is to understand the meaning of implication. Another step is to be able to identify where and how the hypothesis and conclusion are used in the proof. (Several exercise sections have questions that lead the student through this process.) It is expected that students will study the proofs in the text as "worked examples" and will become cognizant of direct and indirect proofs. It is also expected that students will be able to complete a proof given an outline with extensive hints. Several homework exercises are of this nature.

The length of the first chapter (Functions) requires an explanation. Since a major objective of the text is to help guide students in learning how to analyze functions, it is essential to establish a firm basis for this analysis at the beginning of the course. The authors have found that the time spent on this chapter pays major dividends as the course unfolds.

The lengths of sections is determined by the concept being discussed and not by the amount of material that can be presented in one class. Thus several sections (may) require more than one class period. This feature of the text provides flexibility for the instructor to "tailor" the material to an individual class by determining the "class breaks" rather than relying on the authors doing it. This is particularly important in the first chapter, considering the intended audience. The chapter contains many examples; thus the instructor can vary the pace depending upon the backgrounds of the students.

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To the Student

This text is primarily concerned with the study and analysis of functions. Questions such as the following will motivate and guide our work.

- What is a function, f?
- What are the source and target, domain and range of f?
- What does it mean for a function to converge to a limit?
- Is f continuous?
- Is f differentiable? If so, how do we differentiate it?
- Is f integrable? If it is, how do we integrate it?
- For what values of x is $f(x) = 0$?
- How can $f(x)$ be computed for a given x?
- How can a function be approximated, if not computed exactly?
- How can a function be expressed as a series?

The emphasis will be on the process of developing concepts and their extensions to higher dimensions rather than on techniques. Reasoning rather than computation is expected. Sequences and the Basic Approximation Process will be our most important tools. A student successfully completing a course based on this text will be well equipped to pursue additional study in mathematics either independently or in formal courses. More importantly, the student should be able to effectively apply the Basic Approximation Process in problem solving and in developing concepts in other areas of mathematics as well as in related fields: physics, engineering, chemistry, biology, and economics.

An inquisitive attitude is fostered through numerous questions and exercises interspersed throughout the text. Such an attitude is a key for developing and applying mathematics. Effective problem solving approaches consist of asking and answering questions (e.g., What is the question asking?, What do I know?, What do I need to know?). In a broader and more fundamental sense, in the views of the authors, an inquisitive attitude is a necessary part of being an educated person.

A strong emphasis on making up examples are basic ingredients in the drive for conceptual understanding, for problem solving, and for establishing a basis for logical arguments.

Students will find the major concepts (functions, convergence, continuity, differentiation, integration, series representation) developed in this text appearing and reappearing in future studies. For example, convergence and approximation are the "backbones" of analysis courses (e.g., differential equations, complex variables, real analysis). Differentiation and integration are the concepts needed to study change, the source of most mathematical applications. Power series (e.g. geometric series) are used in practically every undergraduate mathematics course.

Clear instructions and models of how to develop and extend concepts along with a firm founding in the basic calculus concepts is the legacy this text leaves to the student.

Notation and Numbering Used in the Text

Theorems, definitions, examples, questions, and figures are numbered consecutively within sections. For example, definition 1.2.3 will be in chapter 1 section 2 immediately before example 1.2.4. The end of examples, proofs, and questions are indicated by $\triangle$, $\Box$, $\Diamond$respectively. Theorems and definitions are set in italics to emphasize their importance. Standard functions, such as sin and log are set in standard type, while other functions, such as f, are set in italics.

Questions should be worked by the student as they are encountered as a check on understanding. Completely worked answers to the questions appear in the Appendix.

Exercises appear at the end of each section. They are often in odd/even pairs, with the answer to the odd numbered exercises given in the Appendix. Exercises involving the use of a calculator or computer are marked with a "microcomputer" icon in the margin.

Don Small
John Hosack

Post Script

We are, in fact, giving a dishonest picture of Mathematics if we do not allow the student to participate in finding the right problem or theorem. It has often been said that, once a mathematician knows what he is trying to prove, his job is half over. This may be an over simplification; nevertheless, the normal state of mathematical activity is one that involves (a) a situation that is crying out for understanding, and (b) a search for the right way to look at it. Unfortunately, we often exclude this intuitive discovery aspect of mathematics from our teaching. A carefully organized course in mathematics is sometimes too much like a hiking trip in the mountains that never leaves the well-worn trails. The tour manages to visit a steady sequence of the "high spots" of the natural scenery. It carefully avoids all false starts, dead-ends, and impossible barriers, and arrives by five o'clock every evening at a well-stocked cabin. The order of difficulty is carefully controlled, and it is obviously a most pleasant way to proceed. However, the hiker misses the excitement of risking an enforced camping out, of helping locate a trail, and of making his way cross-country with only intuition and a compass as a guide. "Cross-country" mathematics is a necessary ingredient of a good education.

Henry Pollak

Contents