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Abstract

This paper deals with elliptic equations in the plane with degeneracies. The equations are generated by a complex vector field that is elliptic everywhere except along a simple closed curve. Kernels for these equations are constructed. Properties of solutions, in a neighborhood of the degeneracy curve, are obtained through integral and series representations. An application to a second order elliptic equation with a punctual singularity is given.

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Contents

Introduction

This paper deals with the properties of solutions of first and second order equations in the plane. These equations are generated by a complex vector field X that is elliptic everywhere except along a simple closed curve $\Sigma \subset \mathbb{R}^2$. The vector field X is tangent to Σ and $X \wedge \overline{X}$ vanishes to first order along Σ (and so X does not satisfy Hörmander's bracket condition). Such vector fields have canonical representatives (see [**8**]). More precisely, there is a change of coordinates in a tubular neighborhood of Σ such that X is conjugate to a unique vector field L of the form

$$L = \lambda \frac{\partial}{\partial t} - ir \frac{\partial}{\partial r} \tag{0.1}$$

defined in a neighborhood of the circle $r = 0$ in $\mathbb{R} \times \mathbb{S}^1$, where $\lambda \in \mathbb{R}^+ + i\mathbb{R}$ is an invariant of the structure generated by X. We should point out that normalizations for vector fields X such that $X \wedge \overline{X}$ vanishes to a constant order $n > 1$ along Σ are obtained in [**9**], but we will consider here only the case $n = 1$. This canonical representation makes it possible to study the equations generated by X in a neighborhood of the characteristic curve Σ. We would like to mention a very recent paper by F. Treves [**13**] that uses this normalization to study hypoellipticity and local solvability of complex vector fields in the plane near a linear singularity. The motivation for our work stems from the theory of hypoanalytic structures (see [**12**] and the references therein) and from the theory of generalized analytic functions (see [**18**]).

The equations considered here are of the form

$$Lu = F(r, t, u) \quad \text{and} \quad Pu = G(r, t, u, Lu),$$

where P is the (real) second order operator

$$P = L\overline{L} + \beta(t)L + \overline{\beta}(t)\overline{L}\,. \tag{0.2}$$

It should be noted that very little is known, even locally, about the structure of the solutions of second order equations generated by complex vector fields with degeneracies. The paper [**5**] explores the local solvability of a particular case generated by a vector field of finite type.

An application to a class of second order elliptic operators with a punctual singularity in $\mathbb{R}^2$ is given. This class consists of operators of the form

$$D = a_{11}\frac{\partial^2}{\partial x^2} + 2a_{12}\frac{\partial^2}{\partial xy} + a_{22}\frac{\partial^2}{\partial y^2} + a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y}\,, \tag{0.3}$$

where the coefficients are real-valued, smooth, vanish at 0, and satisfy

$$C_1 \le \frac{a_{11}a_{22} - a_{12}^2}{(x^2+y^2)^2} \le C_2$$

for some positive constants $C_1 \le C_2$. It turns out that each such operator D is conjugate in $U\backslash 0$ (where U is an open neighborhood of $0 \in \mathbb{R}^2$) to a multiple of an operator P given by (0.2).

Our approach is based on a thorough study of the operator $\mathcal{L}$ given by

$$\mathcal{L}u = Lu - A(t)u - B(t)\overline{u}\,. \tag{0.4}$$

For the equation $\mathcal{L}u = 0$, we introduce particular solutions, called here basic solutions. They have the form

$$w(r,t) = r^{\sigma}\phi(t) + \overline{r^{\sigma}\psi(t)}\,,$$

where $\sigma \in \mathbb{C}$ and $\phi(t)$, $\psi(t)$ are 2π-periodic and $\mathbb{C}$-valued. Chapters 2 and 4 establish the main properties of the basic solutions. In particular, we show that for every $j \in \mathbb{Z}$, there are (up to real multiples) exactly two $\mathbb{R}$-independent basic solutions

$$w_j^{\pm}(r,t) = r^{\sigma_j^{\pm}}\phi_j^{\pm}(t) + \overline{r^{\sigma_j^{\pm}}\psi_j^{\pm}(t)}$$

with winding number j. For a given j, if $\sigma_j^+ \in \mathbb{C}\backslash\mathbb{R}$, then $\sigma_j^- = \sigma_j^+$; and if $\sigma_j^+ \in \mathbb{R}$ then we have only $\sigma_j^- \le \sigma_j^+$. The basic solutions play a fundamental role in the structure of the space of solutions of the equation $\mathcal{L}u = F$.

In Chapter 6, we show that any solution of $\mathcal{L}u = 0$ in a cylinder $(0,\ R) \times \mathbb{S}^1$ has a Laurent type series expansion in the $w_j^{\pm}$'s. From the basic solutions of $\mathcal{L}$ and those of the adjoint operator $\mathcal{L}^*$, we construct, in Chapter 5, kernels Ω_1 and Ω_2 that allow us to obtain a Cauchy Integral Formula (Chapter 6)

$$u(r,t) = \int_{\partial_0 U} \Omega_1 u\,\frac{d\zeta}{\zeta} + \overline{\Omega_2 u}\,\frac{d\overline{\zeta}}{\overline{\zeta}} \tag{0.5}$$

that represents the solution u of $\mathcal{L}u = 0$ in terms of its values on the distinguished boundary $\partial_0 U = \partial U\backslash\Sigma$.

For the nonhomogeneous equation $\mathcal{L}u = F$, we construct, in Chapter 7, an integral operator T, given by

$$TF = \frac{-1}{2\pi}\iint_U \left(\Omega_1 F + \overline{\Omega_2 F}\right)\frac{d\rho d\theta}{\rho}\,. \tag{0.6}$$

This operator produces Hölder continuous solutions (up to the characteristic circle Σ), when F is in an appropriate L^p-space. The properties of T allow us to establish, in Chapter 8, a similarity principle between the solutions of the homogeneous equations $\mathcal{L}u = 0$ and those of a semilinear equation $\mathcal{L}u = F(r,t,u)$

The properties of the (real-valued) solutions of $Pu = G$ are studied in Chapters 9 to 11. To each function u we associate a complex valued function $w = BLu$, called here the L-gradient of u, and such that w solves an equation of the form $\mathcal{L}w = F$. The properties of the solutions of $Pu = G$ can thus be understood in terms of the properties of their L-gradients. In particular series representations and integral representations are obtained for u. A maximum principle for the solutions of $Pu = 0$ holds on the distinguished boundary $\partial_0 U$, if the spectral values $\sigma_j^{\pm}$ satisfy a certain condition. In the last chapter, we establish the conjugacy between the operator D and the operator P.

CHAPTER 1

Preliminaries

We start by reducing the main equation $Lu = Au + B\overline{u}$ into a simpler form. Then, we define a family of operators $\mathcal{L}_\epsilon$, their adjoint $\mathcal{L}_\epsilon^*$, and prove a Green's formula. The operators $\mathcal{L}_\epsilon$ will be extensively used in the next chapter.

Let $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}^*$ and define the vector field L by

$$L = \lambda \frac{\partial}{\partial t} - ir\frac{\partial}{\partial r}\,. \tag{1.1}$$

For $A \in C^k(\mathbb{S}^1, \mathbb{C})$, with $k \in \mathbb{Z}^+$, set

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} A(t)dt, \qquad \nu = 1 - \mathrm{Im}\frac{A_0}{\lambda} + \left[\mathrm{Im}\frac{A_0}{\lambda}\right]$$

where for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less or equal than x. Hence, $\nu \in [0,\ 1)$. Define the function

$$m(t) = \exp\left(it + i\left[\mathrm{Im}\frac{A_0}{\lambda}\right]t + \frac{1}{\lambda}\int_0^t (A(s) - A_0)ds\right).$$

Note that $m(t)$ is 2π-periodic. The following lemma is easily verified.

LEMMA 1.1. *Let $A, B \in C^k(\mathbb{S}^1, \mathbb{C})$ and $m(t)$ be as above. If $u(r,t)$ is a solution of the equation*

$$Lu = A(t)u + B(t)\overline{u} \tag{1.2}$$

then the function $w(r,t) = \dfrac{u(r,t)}{m(t)}$ solves the equation

$$Lw = \lambda\left(\mathrm{Re}\frac{A_0}{\lambda} - i\nu\right)w + C(t)\overline{w} \tag{1.3}$$

where $C(t) = B(t)\dfrac{\overline{m}(t)}{m(t)}$.

In view of this lemma, from now on, we will assume that $\mathrm{Re}\dfrac{A_0}{\lambda} = 0$ and deal with the simplified equation

$$Lu = -i\lambda\nu u + c(t)\overline{u} \tag{1.4}$$

where $\nu \in [0,1)$ and $c(t) \in C^k(\mathbb{S}^1, \mathbb{C})$.

Consider the family of vector fields

$$L_\epsilon = \lambda_\epsilon\frac{\partial}{\partial t} - ir\frac{\partial}{\partial r} \tag{1.5}$$

where $\lambda_\epsilon = a + ib\epsilon$, $\epsilon \in \mathbb{R}$, and the associated operators $\mathcal{L}_\epsilon$ defined by

$$\mathcal{L}_\epsilon u(r,t) = \lambda_\epsilon \frac{\partial u}{\partial t}(r,t) - ir\frac{\partial u}{\partial r}(r,t) + i\lambda_\epsilon \nu u(r,t) - c(t)\overline{u(r,t)} \tag{1.6}$$

For $\mathbb{C}$-valued functions defined on an open set $U \in \mathbb{R}^+ \times \mathbb{S}^1$, we define the bilinear form

$$< f, g >= \mathrm{Re}\left(\iint_U f(r,t)g(r,t)\frac{drdt}{r}\right).$$

For the duality induced by this form, the adjoint of $\mathcal{L}_\epsilon$ is

$$\mathcal{L}_\epsilon^* v(r,t) = -\left(\lambda_\epsilon \frac{\partial v}{\partial t}(r,t) - ir\frac{\partial v}{\partial r}(r,t) - i\lambda_\epsilon \nu v(r,t) + \overline{c(t)}\,\overline{v(r,t)}\right) \tag{1.7}$$

The function $z_\epsilon(r,t) = |r|^{\lambda_\epsilon} e^{it}$ is a first integral of L_ϵ in $\mathbb{R}^* \times \mathbb{S}^1$. That is, $L_\epsilon z_\epsilon = 0$, $dz_\epsilon \neq 0$. Furthermore $z_\epsilon : R^+ \times \mathbb{S}^1 \longrightarrow \mathbb{C}^*$ is a diffeomorphism. The following Green's identity will be used throughout.

PROPOSITION 1.2. *Let $U \subset \mathbb{R}^+ \times \mathbb{S}^1$ be an open set with piecewise smooth boundary. Let $u, v \in C^0(\overline{U})$ with $L_\epsilon u$ and $L_\epsilon v$ integrable. Then,*

$$\mathrm{Re}\left(\int_{\partial U} uv\frac{dz_\epsilon}{z_\epsilon}\right) =< u, \mathcal{L}_\epsilon^* v > - < \mathcal{L}_\epsilon u, v > . \tag{1.8}$$

PROOF. Note that for a differentiable function $f(r,t)$, we have

$$df = \frac{i}{2a}\left(-\overline{L_\epsilon} f\frac{dz_\epsilon}{z_\epsilon} + L_\epsilon f\frac{d\overline{z_\epsilon}}{\overline{z_\epsilon}}\right) \quad \text{and} \quad \frac{d\overline{z_\epsilon}}{\overline{z_\epsilon}} \wedge \frac{dz_\epsilon}{z_\epsilon} = \frac{2ia}{r} dr \wedge dt.$$

Hence,

$$\begin{aligned}\int_{\partial U} uv\frac{dz_\epsilon}{z_\epsilon} &= \iint_U \frac{i}{2a}\left(uL_\epsilon v + vL_\epsilon u\right)\frac{d\overline{z_\epsilon}}{\overline{z_\epsilon}} \wedge \frac{dz_\epsilon}{z_\epsilon} \\ &= -\iint_U \left(v\mathcal{L}_\epsilon u - u\mathcal{L}_\epsilon^* v + cv\overline{u} - u\overline{cv}\right)\frac{drdt}{r}.\end{aligned}$$

By taking the real parts, we get (1.8). □

REMARK 1.3. When $b = 0$ so that $\lambda = a \in \mathbb{R}^+$. The pushforward via the first integral $r^a e^{it}$ reduces the equation $\mathcal{L}u = F$ into a Cauchy Riemann equation with a singular point of the form

$$\frac{\partial W}{\partial \overline{z}} = \frac{a_0}{\overline{z}} W + \frac{B(t)}{\overline{z}}\overline{W} + G(z). \tag{1.9}$$

Properties of the solutions of such equations are thoroughly studied in [**10**]. Many aspects of CR equations with punctual singularities have been studied by a number of authors and we would like to mention in particular the following papers [**1**], [**14**], [**15**], [**16**] and [**17**].

REMARK 1.4. We should point out that the vector fields involved here satisfy the Nirenberg-Treves Condition (P) at each point of the characteristic circle. For vector fields X satisfying condition (P), there is a rich history for the local solvability of the $\mathbb{C}$-linear equation $Xu = F$ (see the books [**3**], [**12**] and the references therein). In [**7**], the semiglobal solvability of the equation $Pu = f$ is addressed, where P is a pseudo-differential operator satisfying the Nirenberg-Treves Condition (P). Our focus here is first, on the semiglobal solvability in a tubular neighborhood of the characteristic circle, and second, on the equations containing the term $\overline{u}$ which makes them not $\mathbb{C}$-linear.

REMARK 1.5. The operator $\mathcal{L}_\epsilon$ is invariant under the diffeomorphism $\Phi(r,t) = (-r,t)$ from $\mathbb{R}^+ \times \mathbb{S}^1$ to $\mathbb{R}^- \times \mathbb{S}^1$. Hence, all the results about $\mathcal{L}_\epsilon$ stated in domains contained in $\mathbb{R}^+ \times \mathbb{S}^1$ have their counterparts for domains in $\mathbb{R}^- \times \mathbb{S}^1$. Throughout this paper, we will be mainly stating results for $r \geq 0$.

CHAPTER 2

Basic Solutions

In this section we introduce the notion of basic solutions for $\mathcal{L}_\epsilon$. We say that w is a basic solution of $\mathcal{L}_\epsilon$ if it is a nontrivial solution of $\mathcal{L}_\epsilon w = 0$, in $\mathbb{R}^+ \times \mathbb{S}^1$, of the form

$$w(r,t) = r^\sigma \phi(t) + \overline{r^\sigma \psi(t)}, \tag{2.1}$$

with $\sigma \in \mathbb{C}$ and where $\phi(t), \psi(t)$ are 2π-periodic functions. These solutions play a crucial role for the equations generated by L_ϵ. In a sense, they play roles similar to those played by the functions z^n in classical complex and harmonic analysis.

Consider, as our starting point, the basic solutions of $\mathcal{L}_0$. These basic solutions are known, since they can be recovered from those of equation (1.9) (see Remark 1.1). From $\mathcal{L}_0$, we obtain the properties of the basic solutions for $\mathcal{L}_\epsilon$. This is done through continuity arguments in the study of an associated system of 2×2 ordinary differential equations in $\mathbb{C}^2$ with periodic coefficients. By using analytic dependence of the system with respect to the parameters, the spectral values σ of the monodromy matrix can be tracked down. The main result (Theorem 2.1) states that for every $j \in \mathbb{Z}$, the operator $\mathcal{L}_\epsilon$ has exactly two $\mathbb{R}$-independent basic solutions with winding number j.

2.1. Properties of basic solutions

We prove that a basic solution has no vanishing points when $r > 0$ and that one of its components ϕ or ψ is always dominating.

It is immediate, from (1.6), that in order for a function $w(r,t)$, given by (2.1), to satisfy $\mathcal{L}_\epsilon w = 0$, the components ϕ and ψ need to be periodic solutions of the system of ordinary differential equations

$$\begin{cases} \lambda_\epsilon \phi'(t) = i(\sigma - \lambda_\epsilon \nu)\phi(t) + c(t)\psi(t) \\ \overline{\lambda_\epsilon}\psi'(t) = -i(\sigma - \overline{\lambda_\epsilon}\nu)\psi(t) + \overline{c(t)}\phi(t) \,. \end{cases} \tag{2.2}$$

Note that if $\sigma \in \mathbb{R}$, then $w = r^\sigma(\phi(t) + \overline{\psi(t)})$ and $f = \phi + \overline{\psi}$ solves the equation

$$\lambda_\epsilon f'(t) = i(\sigma - \lambda_\epsilon \nu) f(t) + c(t)\overline{f(t)} \,. \tag{2.3}$$

Now we prove that a basic solution cannot have zeros when $r > 0$.

PROPOSITION 2.1. *Let $w(r,t)$, given by* (2.1), *be a basic solution of $\mathcal{L}_\epsilon$. Then*

$$w(r,t) \neq 0 \qquad \forall (r,t) \in \mathbb{R}^+ \times \mathbb{S}^1 \,.$$

PROOF. If $\sigma \in \mathbb{R}$, we have $w(r,t) = r^\sigma f(t)$ with $f(t)$ satisfying (2.3). If $w(r_0, t_0) = 0$ for some $r_0 > 0$, then $f(t_0) = 0$ and so $f \equiv 0$ by uniqueness of solutions of the differential equation (2.3). Now, assume that $\sigma = \alpha + i\beta$ with $\beta \in \mathbb{R}^*$.

Suppose that w is a basic solution and $w(r_0, t_0) = 0$ for some $(r_0, t_0) \in \mathbb{R}^+ \times \mathbb{S}^1$. Consider the sequence of real numbers $r_k = r_0 \exp(-k\pi/|\beta|)$ with $k \in \mathbb{Z}^+$. Then $r_k \longrightarrow 0$ as $k \longrightarrow \infty$ and $r_k^{2i\beta} = r_0^{2i\beta}$. It follows at once from $w(r_0, t_0) = 0$ and (2.1) that $w(r_k, t_0) = 0$ for every $k \in \mathbb{Z}^+$. Note that from (2.1) we have $|w(r, t)| \le Er^a$, where $E = \max(|\phi(t)| + |\psi(t)|)$. Note also that since $\mathcal{L}_\epsilon$ is elliptic in $\mathbb{R}^+ \times \mathbb{S}^1$, then the zeros of any solution of the equation $\mathcal{L}_\epsilon u = 0$ are isolated in $\mathbb{R}^+ \times \mathbb{S}^1$.

The pushforward via the mapping $z = r^{\lambda_\epsilon} e^{it}$ of the equation $\mathcal{L}_\epsilon w = 0$ in $\mathbb{R}^+ \times \mathbb{S}^1$ is the singular CR equation

$$\frac{\partial W}{\partial \overline{z}} = \frac{\lambda_\epsilon \nu e^{2i\theta}}{2az} W - \frac{C(z) e^{2i\theta}}{2iaz} \overline{W}$$

where $W(z)$ and $C(z)$ are the pushforwards of $w(r, t)$ and $c(t)$ and where θ is the argument of z. We are going to show that W has the form $W(z) = H(z) \exp(S(z))$ where H is holomorphic in the punctured disc $D^*(0, R)$, $S(z)$ continuous in $D^*(0, R)$ and satisfies the growth condition $|S(z)| \le \log \dfrac{K}{|z|^p}$ for some positive constants K and p. For this, consider the function $M(z)$ defined by

$$M(z) = \frac{\lambda_\epsilon \nu e^{2i\theta}}{2a} - \frac{C(z) e^{2i\theta}}{2ia} \frac{\overline{W(z)}}{W(z)}$$

for $0 < |z| < R$, $W(z) \ne 0$ and by $M(z) = 1$ on the set of isolated points where $W(z) = 0$. This function is bounded and it follows from the classical theory of CR equations (see [**2**] or [**18**]) that

$$N(z) = \frac{-1}{\pi} \iint_{D(0,R)} \frac{M(\zeta)}{\zeta - z} d\xi d\eta$$

$(\zeta = \xi + i\eta)$ is continuous, satisfies $\dfrac{\partial N(z)}{\partial \overline{z}} = M(z)$ and

$$|N(z_1) - N(z_2)| \le A||M||_\infty |z_1 - z_2| \log \frac{2R}{|z_1 - z_2|} \quad \forall z_1, z_2 \in D(0, R)$$

for some positive constant A. Define S by $S(z) = \dfrac{N(z) - N(0)}{z}$. We have then, for $z \ne 0$,

$$\frac{\partial S}{\partial \overline{z}} = \frac{W_{\overline{z}}(z)}{W(z)} \quad \text{and} \quad |S(z)| \le B \log \frac{2R}{|z|},$$

with $B = A||M||_\infty$. Let $H(z) = W(z) \exp(-S(z))$. Then H is holomorphic in $0 < |z| < R$ and it satisfies

$$|H(z)| \le |W(z)| \exp(|S(z)|) \le |W(z)| \frac{(2R)^B}{|Z|^B} \le C_1 |z|^s$$

for some constants C_1 and $s \in R$. The last inequality follows from the estimate $|w| \le Er^\alpha$. This means that the function H has at most a pole at $z = 0$. Since $w(r_k, t_0) = 0$, then $H(z_k) = 0$ for every k and $z_k = r_k^{\lambda_\epsilon} e^{it_0} \longrightarrow 0$. Hence $H \equiv 0$ and $w \equiv 0$ which is a contradiction. □

COROLLARY 2.2. *If $w = r^\sigma \phi(t) + \overline{r^\sigma \psi(t)}$ is a basic solution of $\mathcal{L}_\epsilon$ with $\sigma = \alpha + i\beta$ and $\beta \ne 0$, then for every $t \in \mathbb{R}$, $|\phi(t)| \ne |\psi(t)|$.*

PROOF. By contradiction, suppose that there is $t_0 \in \mathbb{R}$ such that $|\phi(t_0)| = |\psi(t_0)|$. Let $x_0 \in \mathbb{R}$ such that $\overline{\psi(t_0)} = -e^{ix_0}\phi(t_0)$. Then the positive number $r_0 = \exp(x_0/2\beta)$ satisfies $r_0^{i\beta} = r_0^{-i\beta}e^{ix_0}$ and consequently,

$$w(r_0, t_0) = r_0^{\alpha}(r_0^{i\beta}\phi(t_0) + \overline{r_0^{i\beta}\psi(t_0)}) = 0.$$

This contradicts Proposition 2.1. □

This corollary implies that, for a given basic solution $w = r^{\sigma}\phi + \overline{r^{\sigma}\psi}$ with $\sigma \in \mathbb{C}\backslash\mathbb{R}$, one of the functions ϕ or ψ is dominant. That is, $|\phi(t)| > |\psi(t)|$ or $|\psi(t)| > |\phi(t)|$ for every $t \in \mathbb{R}$. Hence the winding number of w, $\mathrm{Ind}(w)$ is well defined and we have $\mathrm{Ind}(w) = \mathrm{Ind}(\phi)$ if $|\phi| > |\psi|$ and $\mathrm{Ind}(w) = \mathrm{Ind}(\overline{\psi})$ otherwise. When $\sigma \in \mathbb{R}$, we have $w = r^{\sigma}f(t)$ with f nowhere 0 and so $\mathrm{Ind}(w) = \mathrm{Ind}(f)$.

For a basic solution $w = r^{\sigma}\phi + \overline{r^{\sigma}\psi}$ with $|\phi| > |\psi|$, we will refer to σ as the exponent of w (or a spectral value of $\mathcal{L}_{\epsilon}$) and define the character of w by

$$\mathrm{Char}(w) = (\sigma, \mathrm{Ind}(w)).$$

We will denote by $\mathrm{Spec}(\mathcal{L}_{\epsilon})$ the set of exponents of basic solutions. That is,

$$\mathrm{Spec}(\mathcal{L}_{\epsilon}) = \{\sigma \in \mathbb{C};\ \exists w,\ \mathrm{Char}(w) = (\sigma, \mathrm{Ind}(w))\},. \tag{2.4}$$

REMARK 2.3. When $\sigma \in \mathbb{C}\backslash\mathbb{R}$ and $w = r^{\sigma}\phi(t) + \overline{r^{\sigma}\psi(t)}$ is a basic solution with $\mathrm{Char}(w) = (\sigma, \mathrm{Ind}(\phi))$, the function $\widetilde{w} = r^{\sigma}(i\phi(t)) + \overline{r^{\sigma}i\psi(t)}$ is also a basic solution with $\mathrm{Char}(w) = \mathrm{Char}(\widetilde{w})$ and w, $\widetilde{w}$ are $\mathbb{R}$-independent.

When $\sigma = \tau \in \mathbb{R}$, and $w = r^{\tau}f(t)$ is a basic solution with $\mathrm{Char}(w) = (\tau, \mathrm{Ind}(f))$, it is not always the case that there is a second $\mathbb{R}$-independent basic solution with the same exponent τ. There is however a second $\mathbb{R}$-independent basic solution $\widetilde{w} = r^{\tau'}g(t)$ with the same winding number ($\mathrm{Ind}(f) = \mathrm{Ind}(g)$) but with a different exponent τ' (see Proposition 2.6).

The following proposition follows from the constancy of the winding number under continuous deformations.

PROPOSITION 2.4. *Let* $w_{\epsilon}(r,t) = r^{\sigma(\epsilon)}\phi(t,\epsilon) + \overline{r^{\sigma(\epsilon)}\psi(t,\epsilon)}$ *be a continuous family of basic solutions of* $\mathcal{L}_{\epsilon}$ *with* $\epsilon \in I$, *where* $I \subset \mathbb{R}$ *is an interval. Then* $\mathrm{Char}(w_{\epsilon})$ *depends continuously on* ϵ *and* $\mathrm{Ind}(w_{\epsilon})$ *is constant.*

2.2. The spectral equation and $\mathrm{Spec}(\mathcal{L}_0)$

We use the 2×2 system of ordinary differential equations to obtain an equation for the spectral values in terms of the monodromy matrix. Results about the CR equation (1.9) are then used to list the properties of $\mathrm{Spec}(\mathcal{L}_0)$.

In order for a function

$$w(r,t) = r^{\sigma}\phi(t) + \overline{r^{\sigma}\psi(t)}$$

to be a basic solution of $\mathcal{L}_{\epsilon}$, the 2π-periodic and $\mathbb{C}^2$-valued function $V(t) = \begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix}$ must solve the periodic system of differential equations

$$\dot{V} = \mathbf{M}(t, \sigma, \epsilon)V \tag{$\mathrm{E}_{\sigma,\epsilon}$}$$