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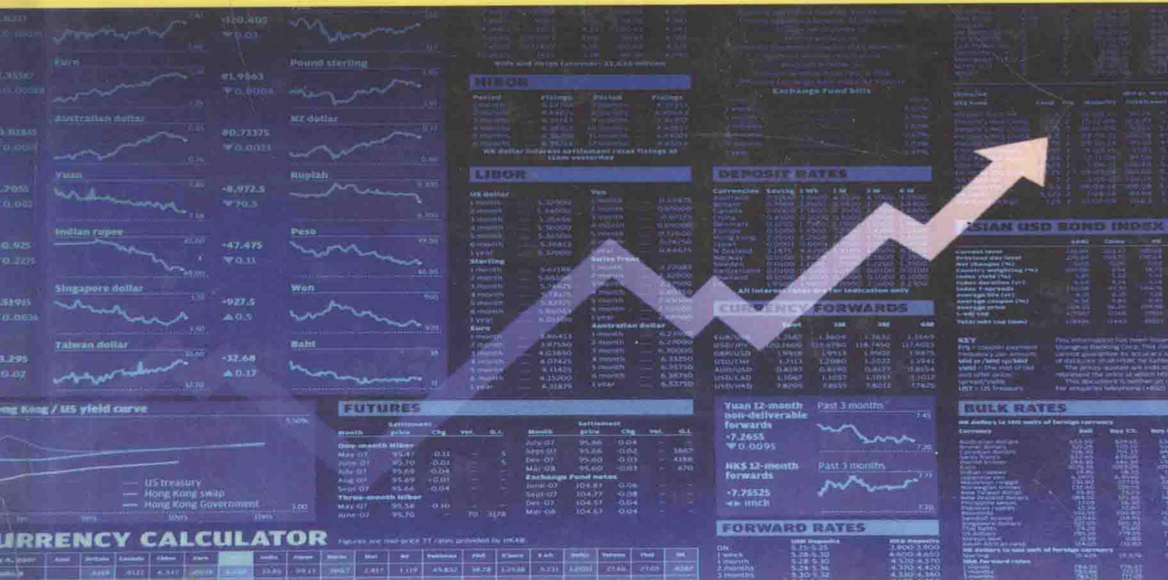
Discrete-time Asset Pricing Models in Applied Stochastic Finance

P.C.G. Vassiliou



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Discrete-time Asset Pricing Models
in Applied Stochastic Finance

To my daughter Olympia, the bravest fighter in life-threatening situations I have ever known.

To my daughter Aglaia, the greatest supporter and joy for all of us.

To my wife Febronia, the most patient and reliable person in my life.

Preface

The revolutionary theory in stochastic finance, which advanced the subject from the nebulous intuitive stage to the level of an exciting and fast growing scientific discipline, was the work of Fischer Black, Myron Scholes and Robert Merton in 1973. Twenty-four years later, in 1997, the Nobel Prize was awarded to the two then living authors, Myron Scholes and Robert Merton.

Stochastic finance and financial engineering sciences have been rapidly expanding over the past four decades. The main reason for this is the development of sophisticated quantitative methodologies that enabled professionals to manage financial risks. In recent years, we have witnessed a tremendous acceleration in research efforts aimed at better comprehension, modeling, and hedging these kinds of risks.

The writing of the present book started seven years ago. Its first version has over the past six years appeared as a basic textbook in an undergraduate and a postgraduate level course for the students in the Mathematics Department, Aristotle University of Thessaloniki, Greece, as well as in other departments with students from various backgrounds. The entire project benefited immensely from the presence of the author as a Visiting Professor in the Department of Statistical Sciences, University College London, for over a year; especially, the advance course given by the author at the London Taught Course Center (LTCC) to the second and third year PhD students of University College London, Imperial College London, Kings College, London School of Economics, and Queen Mary College. The course was a continuous time version of the present book. The response and assessment of the research students greatly helped in completing the project.

This book aims to provide a foundation course on applied stochastic finance. It assumes knowledge of only an introductory course in probability theory and basic mathematical analysis. It is designed for three groups of readers. First, students coming from various backgrounds seeking a basic knowledge on the subject of stochastic finance on which to build in various directions; second, financial analysts and practitioners in the investment, banking, and insurance industries; third, other professionals who are interested in learning, through finance, advanced mathematical and stochastic methods that are basic to many areas of study.

The author intends to (a) take the lecturer's stand in the style of writing, i.e. to motivate the reader for learning and using the various mathematical and stochastic tools through intuitive explanations, by building step by step only the necessary stochastic analysis, stochastic process, and measure theory background, while keeping in mind the final goal of learning the financial techniques. The author feels that finance can be a strong motivational factor and a great way to learn stochastic and mathematics at an advanced level. That in finance pays back enormously; (b) emphasize the clarity of exposition on the generality of results and techniques in order to maximize their efficient use in applications by making the book a useful guide for pricing and hedging any derivative security included in the text or not by including all proofs and/or necessary references; however, the level of detail included is possibly high but without any additional mathematical complexity; (c) provide a pedagogical exposition of stochastic finance methodologies that can be used as teaching material for undergraduate and postgraduate courses for students with various backgrounds. Finally the author makes the effort and takes the time and energy to make available to the average reader a self-teaching book, which is a rare quality of any book, especially one on finance.

The usefulness of this book to students with various backgrounds is facilitated by its organizational design. The book starts with a review of important results from probability theory that will be useful throughout the entire book (recommended also for the experienced reader to refresh some results and probably gain some insight) and is followed by an introduction to the basic financial instruments and the fundamental principles of financial modeling and arbitrage valuation of derivatives. Chapter 3 presents in detail and depth one of the most useful concepts in applied stochastic finance, namely, the concept of conditional expectation. It also offers a compact foundation and presentation of basic results on Markov chains, which play a vital role in modern aspects of stochastic finance. Chapter 4 introduces the discrete time binomial model and uses it among others as a basic pedagogical instrument for presenting in depth all relevant concepts of applied stochastic finance. Chapter 5 builds step by step, enunciated by examples, the most important results from the theory of martingales that are used in the theory and applications of stochastic finance. The days when one could get a good job if one could only spell the world "martingale" are long past. Numerous examples, each highlighted and isolated from the text for easy reference and identification, are introduced. In Chapter 6, at first glance, the stochastic level of the book seems to take a jump. However, the reader is already smoothly prepared to absorb and gain sufficient depth into concepts such as Randon-Nikodým derivative, equivalent martingale measure, non-arbitrage and complete general markets. By the end of the chapter many of the readers will be familiar with most of what is needed from measure theory in stochastic finance. In Chapter 7 we study American derivative securities both using the binomial model and general markets. Chapter 8 is devoted to the study of fixed-income markets and interest rate theory in discrete time. Arbitrage pricing is discussed and financial products such as European derivatives, interest rate swaps, interest rate caps and floors and futures contracts are dealt with. Chapter 9 provides basic knowledge of the vast and important subject of credit risk. In our era, we do not have to really emphasize the importance of the theory of credit risk on our financial stability. We conclude the main theme of the book with the study of the Heath-Jarrow-Morton model for the evolution of forward rate process.

The book also contains two appendices. The first one is a short review on the evolution of stochastic mathematics that changed the financial world. The second is devoted to the separating hyperplane theorem in $\mathbb{R}^n$.

I would like to thank Prof. N. Limnios who first insisted that the present project should be undertaken by me. I would also like to thank the Department of Statistical Sciences, University College London for providing everything necessary for the project. I would especially like to thank Prof. Valerie Isham who made my presence at UCL possible and the Heads of the Department, Prof. Trevor Sweeting and Prof. Tom Fearn, for their support and kindness during the writing of the book.

Finally, I would like to deeply thank my family: Olympia, my daughter, who I discovered is the bravest fighter in life-threatening situations I have ever known; Aglaia, my daughter, the greatest supporter of all of us; last but not the least, my wife Febronia, whose patience and ability to bear with me made the completion of the book possible.

Prof. Panos Vassiliou
January 2010

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Chapter 1

Probability and Random Variables

1.1. Introductory notes

In almost all the interesting problems in finance and especially in those where it is very important to have an answer, there exists a large degree of randomness. In other words, the financial entities that play an important role, for example, the prices of assets, cannot be predicted with accuracy and certainty. The variations that are observed seem to follow the laws of chance. An immediate consequence of this is that the models that need to be constructed for the description and prediction of these important entities contain random variables. Such models are called stochastic models.

The basis for the study of the randomness of various phenomena we observe and describe in nature are probability theory and, what may be thought as its extension, the theory of stochastic processes. In order to provide the reader with as much background as possible in this self-teaching book, we present in this chapter the gist of probability theory, which will be useful in understanding the subjects that follow later. The material is presented in a compact manner, mostly omitting long proofs and elaborations. The readers may either skim through the chapter quickly to refresh their memory or skip it until they need to refer back to something.

It is difficult to include in a moderate chapter an introduction to probability theory for a reader who has not undergone even an introductory course. However, we will try to provide some insight into probability concepts, which will prove useful to readers with all levels of exposure to the subject of probability theory, omitting most of the proofs. This is usual in books especially on stochastic processes. The reader may find useful review chapters in Stirzaker (2005) and Ross (1996, 2007). For those readers who need an accompanying introductory book in addition, Ross (2002) and Stirzaker (2007) are recommended.

The important results from the theory of stochastic processes will also be presented in separate chapters, just before their need arises in the book. An example is the chapter on discrete martingales.

Further, advanced knowledge from probability theory will also be developed and presented in separate chapters just before the chapter where it will be used. An example is the chapter on conditional expectation which precedes the chapter on discrete martingales.

1.2. Probability space

It is common experience in life that there exist situations the direction of evolution of which is not known, i.e. their outcome is not predictable in advance. However, there are cases where while the outcome is not known in advance, the set of all possible outcomes is known. Any activity or procedure that may give rise to a well-defined set of outcomes is called an *experiment*. The set of all possible outcomes of an experiment is called the *sample space*, and is denoted by Ω. A particular but unspecified outcome in Ω may be denoted by ω. Each outcome ω belonging to the sample space Ω is called an *elementary event*, while a subset A of Ω is called an *event*. In particular, Ω is called the *certain event*. For any two events A and B of a sample space Ω we define the new event $A \cup B$, which consists of all outcomes that are either in A or in B or in both A and B. The event $A \cup B$ will be referred to as the *union* of the event A and the event B. For any two events A and B, we also define the new event $A \cap B$ or AB, and refer to it as the *intersection* of A and B as follows: AB consists of all outcomes which are *both* in A and B. If the two events A and B are such that the event AB would not consist of any outcomes, then we write $AB = \varnothing$. We will denote this event as the null event, and A and B are said to be *mutually exclusive*.

We also define unions and intersections of a countable number of events in a similar manner. If $A_1, A_2, ...$ are events, the union of these events, denoted by $\cup_{n=1}^{\infty} E_n$, is defined to be that event, which consists of all outcomes that are in E_n for at least one value of $n = 1, 2, ...$ Similarly, the intersection of the events E_n denoted by $\cap_{n=1}^{\infty} E_n$, is defined to be the event consisting of those outcomes that are in all events E_n, $n = 1, 2, ...$ For any event A we define the new event A^c, referred to as the *complement* of A, to consist of all outcomes in the sample space Ω that are not in A.

EXAMPLE 1.1. If the experiment consists of rolling a dice, then the sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

where the outcome i means that i appeared in the dice. If $A = \{1,3,4\}$ and $B = \{2,3,5\}$ then

$$A \cup B = \{1,2,3,4,5\}, \mathrm{A} \cap \mathrm{B} = \{3\}\ , \mathrm{A}^{\mathrm{c}} = \{2,5,6\}.$$

At this point, we must remark that, while all events are subsets of Ω keep track of events, we require that the family of events is closed under intersections and unions. Formally we make the following definition.

DEFINITION 1.2. A collection of events is called an event space, and denoted by $\mathcal{F}$, if it satisfies

(i) $\varnothing$ is in $\mathcal{F}$;

(ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;

(iii) if $A_n \in \mathcal{F}$ for $n = 1, 2, ...$, then $\cup_{n=1}^{\infty} A_n$ is in $\mathcal{F}$.

It is known from set theory that a collection of sets that satisfy the above is called a σ-algebra or a σ-field.

There are many experiments in which all elementary events of the sample space have the same chance to occur. Our intuition and experience guide us in this case to think that all the elementary events should have equal probabilities to occur. This empirical observation led Abraham De Moivre in 1711 to the following definition:

Let Ω be the sample space of an experiment for which all elementary events are equally probable and let an event $A \subset \Omega$. Then we define the probability of the event A and denote it by $\mathbb{P}(A)$ as follows:

$$\mathbb{P}(A) = \frac{\text{Number of elements of event } A}{\text{Number of elements of sample space } \Omega}.$$

In modern probability theory founded by Kolmogorov (1931), probability is understood to be a set function, defined on $\mathcal{F}$ with values on $[0,1]$, satisfying the following axioms:

(i) $\mathbb{P}(\Omega) = 1$,

(ii) $0 \leq \mathbb{P}(A) \leq 1$ for any event $A \in \mathcal{F}$,

(iii) For mutually exclusive events $A_n \in \mathcal{F}$ for $n = 1, 2, ...$, we have

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n). \qquad (1.1)$$

Note that the physical meaning of probability is not important in this definition. Property (1.1) is called the *σ-additivity.*

Given a sample space Ω and a σ-algebra $\mathcal{F}$ of its subsets, if a set function $\mathbb{P}(.)$ satisfies the above properties, we call $\mathbb{P}(.)$ a *probability measure* (or probability for short). The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.*

In the rest of this chapter, we assume that the σ-algebra $\mathcal{F}$ is rich enough, meaning that all events under consideration belong to $\mathcal{F}$. That is, we consider a probability of an event without mentioning that the event indeed belongs to the σ-algebra.

We have chosen to give a rather formal definition of probability as being a set function on a σ-algebra of events. At this point we should remark that if we repeat our experiment a large number of times then the proportion of times an event A will occur converges to $\mathbb{P}(A)$. Observations of this nature led also to a very interesting axiomatic foundation of probability theory via expectation, i.e. on axioms defining the expectation, see Whittle (2000).

EXAMPLE 1.3. Let us assume that we bought an equity and decided as an experiment to follow its evolution in the next three days. We assign the number 1 if the price of the equity was increased on a certain day, and 0 if the price of the equity was decreased on the same day. The sample space of this experiment is then the following:

$$\Omega = \{(1,1,1), (1,1,0), (1,0,1), (0,1,1), (1,0,0), (0,1,0), (0,0,1), (0,0,0)\}.$$

(a) Let us consider the set of subsets of Ω

$$\mathcal{F}_0 = \{\varnothing, \Omega\}.$$

Then $\mathcal{F}_0$ is a σ-algebra on Ω, since

(i) $\Omega \in \mathcal{F}_0$;

(ii) The complement of $\varnothing$ is $\Omega \in \mathcal{F}_0$ and the complement of Ω is $\varnothing \in \mathcal{F}_0$;

(iii) Apparently,

$$\Omega \cup \varnothing \in \mathcal{F}_0.$$

The σ-algebra $\mathcal{F}_0$ contains no information for the true outcome ω for any day since the set $\varnothing$ and the whole sample space Ω are always resolved, even without any information. The σ-algebra $\mathcal{F}_0$ is called the trivial σ-algebra.

(b) Let us consider the set of subsets of Ω

$$\begin{aligned}\mathcal{F}_1 \;=\; & \{\varnothing, \Omega, [(1,1,1), (1,1,0), (1,0,1), (1,0,0)], \\ & [(0,1,1), (0,1,0), (0,0,1), (0,0,0)]\}.\end{aligned}$$