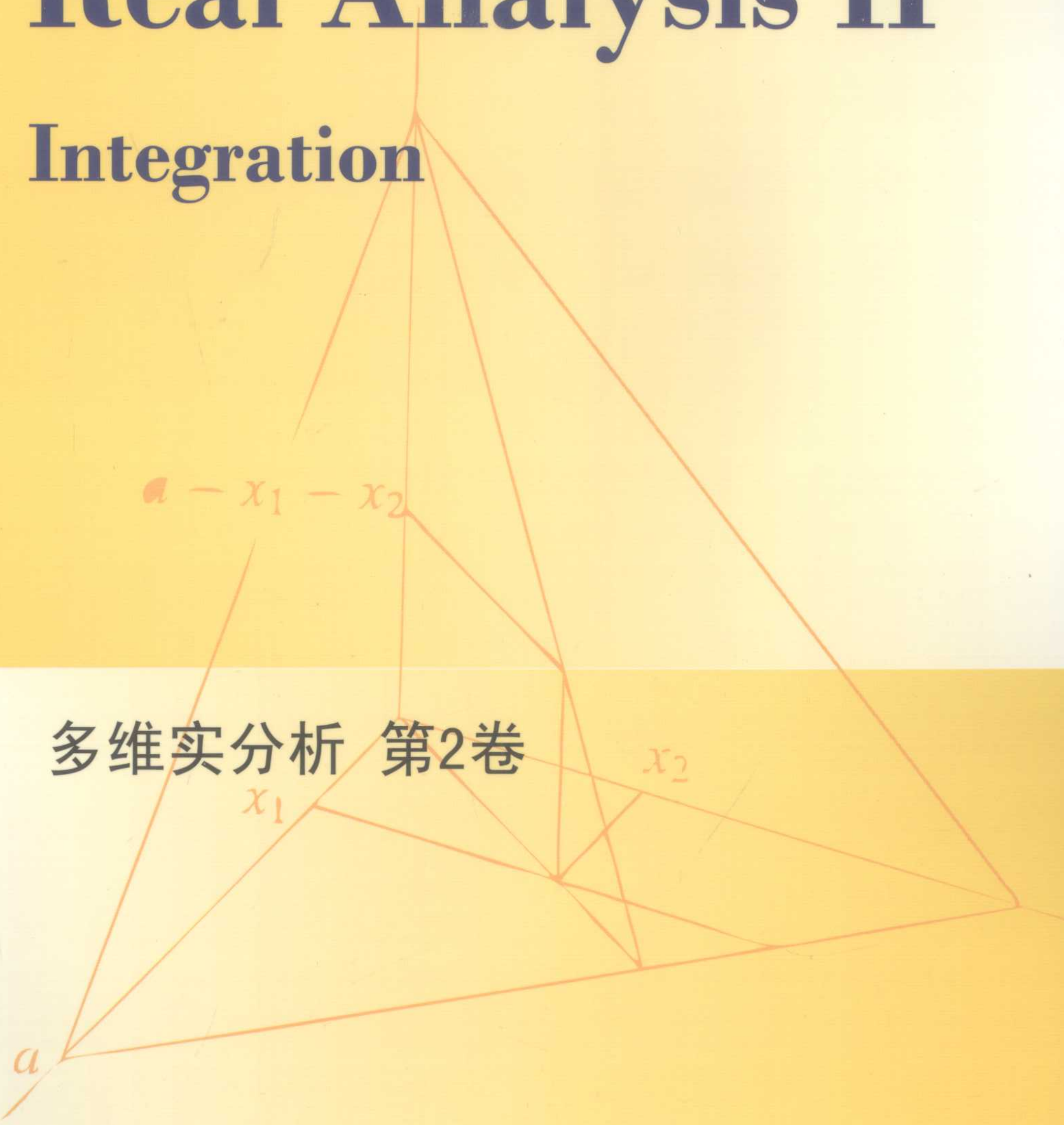
J. J. Duistermaat, J. A. C. Kolk

Multidimensional Real Analysis II

Integration



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MULTIDIMENSIONAL REAL ANALYSIS II:

INTEGRATION

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To Saskia and Floortje

With Gratitude and Love

Preface

I prefer the open landscape under a clear sky with its depth of perspective, where the wealth of sharply defined nearby details gradually fades away towards the horizon.

This book, which is in two parts, provides an introduction to the theory of vector-valued functions on Euclidean space. We focus on four main objects of study and in addition consider the interactions between these. Volume I is devoted to differentiation. Differentiable functions on $\mathbf{R}^n$ come first, in Chapters 1 through 3. Next, differentiable manifolds embedded in $\mathbf{R}^n$ are discussed, in Chapters 4 and 5. In Volume II we take up integration. Chapter 6 deals with the theory of n-dimensional integration over $\mathbf{R}^n$. Finally, in Chapters 7 and 8 lower-dimensional integration over submanifolds of $\mathbf{R}^n$ is developed; particular attention is paid to vector analysis and the theory of differential forms, which are treated independently from each other. Generally speaking, the emphasis is on geometric aspects of analysis rather than on matters belonging to functional analysis.

In presenting the material we have been intentionally concrete, aiming at a thorough understanding of Euclidean space. Once this case is properly understood, it becomes easier to move on to abstract metric spaces or manifolds and to infinite-dimensional function spaces. If the general theory is introduced too soon, the reader might get confused about its relevance and lose motivation. Yet we have tried to organize the book as economically as we could, for instance by making use of linear algebra whenever possible and minimizing the number of ϵ–δ arguments, always without sacrificing rigor. In many cases, a fresh look at old problems, by ourselves and others, led to results or proofs in a form not found in current analysis textbooks. Quite often, similar techniques apply in different parts of mathematics; on the other hand, different techniques may be used to prove the same result. We offer ample illustration of these two principles, in the theory as well as the exercises.

A working knowledge of analysis in one real variable and linear algebra is a prerequisite; furthermore, familiarity with differentiable mappings and submanifolds of $\mathbf{R}^n$, as discussed in volume I, for instance. The main parts of the theory can be used as a text for an introductory course of one semester, as we have been doing for second-year students in Utrecht during the last decade. Sections at the end of many chapters usually contain applications that can be omitted in case of time constraints.

This volume contains 234 exercises, out of a total of 568, offering variations and applications of the main theory, as well as special cases and openings toward applications beyond the scope of this book. Next to routine exercises we tried also to include exercises that represent some mathematical idea. The exercises are independent from each other unless indicated otherwise, and therefore results are sometimes repeated. We have run student seminars based on a selection of the more challenging exercises.

In our experience, interest may be stimulated if from the beginning the student can perceive analysis as a subject intimately connected with many other parts of mathematics and physics: algebra, electromagnetism, geometry, including differential geometry, and topology, Lie groups, mechanics, number theory, partial differential equations, probability, special functions, to name the most important examples. In order to emphasize these relations, many exercises show the way in which results from the aforementioned fields fit in with the present theory; prior knowledge of these subjects is not assumed, however. We hope in this fashion to have created a landscape as preferred by Weyl,[1] thereby contributing to motivation, and facilitating the transition to more advanced treatments and topics.

[1] Weyl, H.: *The Classical Groups.* Princeton University Press, Princeton 1939, p. viii.

of material, especially because there are many people for whom geometrical reasoning is easier and more natural than purely analytic reasoning, and for them an early exposure to geometrical ideas can only be helpful. As a general guide to selecting material, section headings within chapters are printed in two different styles. Fundamental material is marked by **boldface** headings, while more advanced or supplementary topics are marked by ***boldface italics***. All of the last chapter falls into this category. The same convention of type-face distinguishes those exercises which are central to the development of the mathematics from those which are peripheral.

The exercises form an integral part of the book. They are inserted in the middle of the text, and they are designed to be worked when they are first encountered. Usually the text after an exercise will assume that the reader has worked and understood the exercise. The reader who does not have the time to work an exercise should nevertheless read it and try to understand its result. Hints and some solutions will be found at the end of the book.

Background assumed of the reader

Most of this book should be understandable to an advanced undergraduate or beginning graduate student in theoretical physics or applied mathematics. It presupposes reasonable facility with vector calculus, calculus of many variables, matrix algebra (including eigenvectors and determinants), and a little operator theory of the sort one learns in elementary quantum mechanics. The physical applications are drawn from a variety of fields, and not everyone will feel at home with them all. It should be possible to skip many sections on physics without undue loss of continuity, but it would probably be unrealistic to attempt this book without some familiarity with classical mechanics, special relativity, and electromagnetism. The bibliography at the end of chapter 1 lists some books which provide suitable background.

I want to acknowledge my debt to the many people, both colleagues and teachers, who have helped me to appreciate the beauty of differential geometry and understand its usefulness in physics. I am especially indebted to Kip Thorne, Rafael Sorkin, John Friedman, and Frank Estabrook. I also want to thank the first two and many patient students at University College, Cardiff, for their comments on earlier versions of this book. Two of my students, Neil Comins and Brian Wade, deserve special mention for their careful and constructive suggestions. It is also a pleasure to thank Suzanne Ball, Jane Owen, and Margaret Wilkinson for their fast and accurate typing of the manuscript through all its revisions. Finally, I thank my wife for her patience and encouragement, particularly during the last few hectic months.

Cardiff, 30 June 1979 Bernard Schutz

Introduction

Motivation. Analysis came to life in the number space $\mathbf{R}^n$ of dimension n and its complex analog $\mathbf{C}^n$. Developments ever since have consistently shown that further progress and better understanding can be achieved by generalizing the notion of space, for instance to that of a manifold, of a topological vector space, or of a scheme, an algebraic or complex space having infinitesimal neighborhoods, each of these being defined over a field of characteristic which is 0 or positive. The search for unification by continuously reworking old results and blending these with new ones, which is so characteristic of mathematics, nowadays tends to be carried out more and more in these newer contexts, thus bypassing $\mathbf{R}^n$. As a result of this the uninitiated, for whom $\mathbf{R}^n$ is still a difficult object, runs the risk of learning analysis in several real variables in a suboptimal manner. Nevertheless, to quote F. and R. Nevanlinna: "The elimination of coordinates signifies a gain not only in a formal sense. It leads to a greater unity and simplicity in the theory of functions of arbitrarily many variables, the algebraic structure of analysis is clarified, and at the same time the geometric aspects of linear algebra become more prominent, which simplifies one's ability to comprehend the overall structures and promotes the formation of new ideas and methods".[2]

In this text we have tried to strike a balance between the concrete and the abstract: a treatment of integral calculus in the traditional $\mathbf{R}^n$ by efficient methods and using contemporary terminology, providing solid background and adequate preparation for reading more advanced works. The exercises are tightly coordinated with the theory, and most of them have been tried out during practice sessions or exams. Illustrative examples and exercises are offered in order to support and strengthen the reader's intuition.

Organization. This is the second volume, devoted to integration, of a book in two parts; the first volume treats differentiation. The volume at hand uses results from the preceding one, but it should be accessible to the reader who has acquired a working knowledge of differentiable mappings and submanifolds of $\mathbf{R}^n$. Only some of the exercises might require special results from Volume I.

In a subject like this with its many interrelations, the arrangement of the material is more or less determined by the proofs one prefers to or is able to give. Other ways of organizing are possible, but it is our experience that it is not such a simple matter to avoid confusing the reader. In particular, because the Change of Variables Theorem in the present volume is about diffeomorphisms, it is necessary to introduce these initially, in Volume I; a subsequent discussion of the Inverse Function Theorems then is a plausible inference. Next, applications in geometry, to the theory of differentiable manifolds, are natural. This geometry in its turn is indispensable for the description of the boundaries of the open sets that occur in this volume, in the Theorem on Integration of a Total Derivative in $\mathbf{R}^n$, the generalization to $\mathbf{R}^n$ of the

[2] Nevanlinna, F., Nevanlinna, R.: *Absolute Analysis*. Springer-Verlag, Berlin 1973, p. 1.

Fundamental Theorem of Integral Calculus on **R**. This is why differentiation is treated in the first volume and integration in this second. Moreover, most known proofs of the Change of Variables Theorem require an Inverse Function, or the Implicit Function Theorem, as does our first proof. However, for the benefit of those readers who prefer a discussion of integration at an early stage, we have included a second proof of the Change of Variables Theorem by elementary means.

We have stuck to the (admittedly, old-fashioned) theory of Riemann integration. In our department students take a separate course on Lebesgue integration, where its essential role in establishing completeness in many function spaces is carefully discussed. For the topics in this book, however, the Lebesgue integral is not needed and introducing it would cause an overload. In the applications considered, Arzelà's Dominated Convergence Theorem, for which we give a short proof, is an effective alternative for Lebesgue's Dominated Convergence Theorem.

On some technical points. We have tried hard to reduce the number of ϵ–δ arguments, while maintaining a uniform and high level of rigor.

Even for linear coordinate transformations the Change of Variables Theorem is nontrivial, in contrast to the corresponding result in linear algebra. This stems from the fact that in linear algebra the behavior of volume under invertible linear transformations is usually part of the definition of volume. In analysis the notion of volume relies on the Riemann integral, and for the latter only invariance under translations is an immediate consequence of the definition.

The d-dimensional density on a d-dimensional submanifold in $\mathbf{R}^n$ is considered from two complementary points of view. On the one hand, the tangent space of the manifold can be mapped onto $\mathbf{R}^d \simeq \mathbf{R}^d \times \{0_{\mathbf{R}^{n-d}}\} \subset \mathbf{R}^n$ by means of a suitable orthogonal transformation; pulling back the d-volume on $\mathbf{R}^d$ under this mapping one then finds a d-density on the manifold. On the other hand, one can supplement the basis B_d for the tangent space by a set of mutually perpendicular unit vectors all of which are perpendicular to the tangent space, to form a basis B_n for $\mathbf{R}^n$. Next one defines the d-volume of the span of B_d to be the n-volume of the span of B_n (in other words, area equals volume divided by length). Both ways of thinking lead to the same formalism, which unifies the many different formulae that are in use.

Vector analysis should look familiar to students in physics: therefore we have chosen to center on the notion of vector field initially and on that of differential form only later on. Leitmotiv in our treatment of vector analysis is the generalization of the Fundamental Theorem of Integral Calculus on **R** to a theorem on $\mathbf{R}^n$. There are two aspects to the Fundamental Theorem of Integral Calculus on **R**: the existence of an antiderivative for a continuous function; and the equality of the integral of a derivative of a function over an open set with the integral of the function itself over the boundary of that set. By generalizing the former aspect one arrives at the infinitesimal notions in vector analysis, like grad, curl, div; and at Poincaré's Lemma, and its relation with homotopy. Likewise, the latter aspect leads to the global notions, like the integral theorems, and their relations to homology.

This generalization to $\mathbf{R}^n$ begins with the Theorem on Integration of a Total

Derivative, for which an easy proof is offered, by means of a local substitution of variables that flattens the boundary. All other global theorems are reduced to this theorem.

The existence of an antiderivative (or potential) for a vector field on $\mathbf{R}^n$ with $n > 1$ requires integrability conditions to be satisfied. That is, one needs the vanishing of an obstruction against integrability, viz. of Af, twice the anti-adjoint part of the total derivative Df of the vector field f. In $\mathbf{R}^2$ and $\mathbf{R}^3$, Af essentially is the curl of f. Furthermore, Af approximately equals the sum of the values of f at the vertices of a parallelogram, and that sum in turn is a Riemann sum for a line integral of f along that parallelogram. Globalization of this argument leads to a rudimentary form of Stokes' Integral Theorem: a relation between the circulation of f and a surface integral of Af, i.e. an integral of the obstruction.

Vector analysis in $\mathbf{R}^n$ is not a study of partial derivatives of components of vector-valued functions, leading to a coordinate-dependent formulation and a "débauche d'indices". Rather, it is an investigation of these functions and of their total derivatives in their entirety, which is greatly facilitated by linear algebra, especially by the decomposition of the derivative into self-adjoint and anti-adjoint parts using adjoint linear operators.

The definition of positive orientation of a curve is an infinitesimal one. In concrete examples it is often easy to verify whether it is satisfied without an appeal to geometric intuition. The global definition, which is current in many elementary texts, is less rigorous and may lead to cumbersome formulations and/or proofs, of Green's and Stokes' Integral Theorems in particular.

Although formally the theory of differential forms receives an independent treatment, the stage for it is in fact set by much of the preceding material. The main result in the theory is Stokes' Theorem, and the whole discussion aims at proving that theorem at the earliest possible moment. Therefore we have adopted a definition of exterior derivative whereby we achieve this, and the proof of Stokes' Theorem itself is then presented as a direct generalization of the proof of the rudimentary form mentioned previously. The amount of multilinear algebra required for this has been reduced to a minimum. In particular, the general differential k-form is introduced by means of determinants instead of exterior multiplication of forms of lower order, which usually requires a laborious definition.

Exercises. Quite a few of the exercises are used to develop secondary but interesting themes omitted from the main course of lectures for reasons of time, but which often form the transition to more advanced theories. In many cases, exercises are strung together as projects which, step by easy step, lead the reader to important results. In order to set forth the interdependencies that inevitably arise, we begin an exercise by listing the other ones which (in total or in part only) are prerequisites as well as those exercises that use results from the one under discussion. The reader should not feel obliged to completely cover the preliminaries before setting out to work on subsequent exercises; quite often, only some terminology or minor results are required.

Notational conventions. Our notation is fairly standard, yet we mention the following conventions. Although it will often be convenient to write column vectors as row vectors, the reader should remember that all vectors are in fact column vectors, unless specified otherwise. Mappings always have precisely defined domains and images, thus $f : \text{dom}(f) \to \text{im}(f)$, but if we are unable, or do not wish, to specify the domain we write $f : \mathbf{R}^n \supset\!\!\to \mathbf{R}^p$ for a mapping that is well-defined on some subset of $\mathbf{R}^n$ and takes values in $\mathbf{R}^p$. We write $\mathbf{N}_0$ for $\{0\} \cup \mathbf{N}$, $\mathbf{N}_\infty$ for $\mathbf{N} \cup \{\infty\}$, and $\mathbf{R}_+$ for $\{ x \in \mathbf{R} \mid x > 0 \}$. The open interval $\{ x \in \mathbf{R} \mid a < x < b \}$ in $\mathbf{R}$ is denoted by $]\,a, b\,[$ and not by (a, b), in order to avoid confusion with the element $(a, b) \in \mathbf{R}^2$.

Making the notation consistent and transparent is difficult; in particular, every way of designating partial derivatives has its flaws. Whenever possible, we write $D_j f$ for the j-th column in a matrix representation of the total derivative Df of a mapping $f : \mathbf{R}^n \to \mathbf{R}^p$. This leads to expressions like $D_j f_i$ instead of Jacobi's classical $\frac{\partial f_i}{\partial x_j}$, etc. The convention just mentioned has not been applied dogmatically; in the case of special coordinate systems like spherical coordinates, Jacobi's notation is the one of preference. As a further complication, D_j is used by many authors, especially in Fourier theory, for the momentum operator $\frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$.

We use the following dictionary of symbols to indicate the ends of various items:

❑ Proof
❍ Definition
✩ Example

Contents

Volume I

Chapter 6

Integration

In this chapter we extend to $\mathbf{R}^n$ the theory of the Riemann integral from the calculus in one real variable. Principal results are a reduction of n-dimensional integration to successive one-dimensional integrations, and the Change of Variables Theorem. For this fundamental theorem we give three proofs: one in the main text and two in the appendix to this chapter. Important technical tools are the theorems from Chapter 3 and partitions of unity over compact sets. As applications we treat Fourier transformation, i.e. the decomposition of arbitrary functions into periodic ones; and dominated convergence, being a sufficient condition for the interchange of limits and integration.

6.1 Rectangles

Definition 6.1.1. An *n-dimensional rectangle B*, parallel to the coordinate axes, is a subset of $\mathbf{R}^n$ of the form

$$B = \{\, x \in \mathbf{R}^n \mid a_j \le x_j \le b_j \ \ (1 \le j \le n) \,\}, \tag{6.1}$$

where it is assumed that $a_j, b_j \in \mathbf{R}$ and $a_j \le b_j$, for $1 \le j \le n$, compare with Definition 1.8.18.

The *n-dimensional volume* of B, notation $\operatorname{vol}_n(B)$, is defined as

$$\operatorname{vol}_n(B) = \prod_{1 \le j \le n} (b_j - a_j).$$

Note that $\operatorname{vol}_n(B) = 0$ if there exists a j with $a_j = b_j$, that is, if B is contained in an $(n-1)$-dimensional hyperplane in $\mathbf{R}^n$, of the form $\{\, x \in \mathbf{R}^n \mid x_j = a_j \,\}$.

A *partition* of a rectangle B is a finite collection $\mathcal{B} = \{ B_i \mid i \in I \}$ (here I is called the *index set* of $\mathcal{B}$) of n-dimensional rectangles B_i such that

$$B = \bigcup_{i \in I} B_i; \qquad B_i \cap B_j = \emptyset \qquad \text{or} \qquad \mathrm{vol}_n(B_i \cap B_j) = 0 \qquad \text{if} \qquad i \neq j. \tag{6.2}$$

Let $\mathcal{B}$ and $\mathcal{B}'$ be partitions of a rectangle B, then $\mathcal{B}'$ is said to be a *refinement* of $\mathcal{B}$ if for every $B_i \in \mathcal{B}$ the $B'_j \in \mathcal{B}'$ with $B'_j \subset B_i$ form a partition of B_i. ❍

Proposition 6.1.2. *Assume $\{ B_i \mid i \in I \}$ is a partition of a rectangle $B \subset \mathbf{R}^n$. Then*

$$\mathrm{vol}_n(B) = \sum_{i \in I} \mathrm{vol}_n(B_i).$$

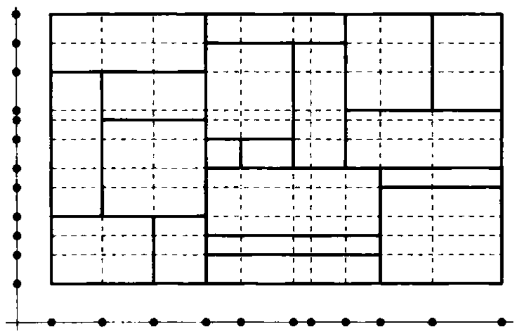

Illustration for the proof of Proposition 6.1.2

Proof. We first prove two auxiliary results.

(i). Assume B as in (6.1); and for $1 \leq j \leq n$, let $t_j \in [a_j, b_j]$ be arbitrary. Consider

$$B' = \{ x \in \mathbf{R}^n \mid a_j \leq x_j \leq t_j, \text{ and } a_k \leq x_k \leq b_k, \text{ for } k \neq j \},$$
$$B'' = \{ x \in \mathbf{R}^n \mid t_j \leq x_j \leq b_j, \text{ and } a_k \leq x_k \leq b_k, \text{ for } k \neq j \}.$$

Because $b_j - a_j = (b_j - t_j) + (t_j - a_j)$, it follows straight away that $\mathrm{vol}_n(B) = \mathrm{vol}_n(B') + \mathrm{vol}_n(B'')$.

(ii). Assume next that for every $1 \leq j \leq n$ the segment $[a_j, b_j]$ is subdivided by the intermediate points

$$a_j = t_j^{(0)} \leq \cdots \leq t_j^{(N(j))} = b_j. \tag{6.3}$$

Then we have, for every n-tuple

$$\alpha = (\alpha(1), \ldots, \alpha(n)) \in \mathbf{N}^n \qquad \text{where} \qquad 1 \leq \alpha(j) \leq N(j), \tag{6.4}$$