Graduate Series in Mathematics **5**

Wang Guorong Wei Yimin Qiao Sanzheng

Generalized Inverses: Theory and Computations

(广义逆：理论与计算)

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Generalized inverses arise in various applications in statistics, science and engineering, such as least squares approximation, singular differential and difference equations,singular control, Markov chains and ill-posed problems. This book contains the latest developments in the theory and computations of generalized inverses.

This book was written for researchers in matrix theory,numerical linear algebra,parallel computations and,particularly,the generalized inverses with applications.And it can also be used as a text or reference for a graduate course.As prerequisites,we assume that reader is with basic linear algebra,matrix theory and functional analysis.

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Preface

The concept of generalized inverses was first introduced by I. Fredholm[57] in 1903, where a generalized inverse of an integral operator was given and was called "pseudoinverse". Generalized inverses of differential operators were implied in D. Hilbert's[78] discussion of generalized Green's functions in 1904. See W. Reid's[131] paper in 1931 for a history of generalized inverses of differential operators.

The generalized inverse of matrices was first introduced by E. H. Moore[113] in 1920, who defined a unique generalized inverse by means of projectors of matrices. Little was done in the next 30 years until mid-1950s when discoveries of the least-squares properties of certain generalized inverses and the relationship of generalized inverses to solutions of linear systems brought new interests in the subject. In particular, R. Penrose[119] showed in 1955 that the Moore's inverse is the unique matrix satisfying four matrix equations. This important discovery revived the study of gene-ralized inverses. In honor of Moore and Penrose's contribution, this unique generalized inverse is called the Moore-Penrose inverse.

The theory, applications and computational methods of generalized inverses have been developing rapidly during the last 50 years. One milestone is the publication of several monographs ([7], [13], [65]and[129]) on the subject in 1970s, particularly, the excellent volume by A. Ben-Israel and T. N.E Greville[7] which has made a long lasting impact on the subject; the other milestone is the publications of two volumes of proceedings. The first is the volume of proceedings[114] of the Advanced Seminar on Generalized Inverses and Applications held at the University of Wisconsin-Madison in 1973 edited by M. Z. Nashed. It is an excellent and extensive survey book. It contains 14 survey papers on the theory, computations and applications of generalized inverses and an exhaustive bibliography that includes all related references up to 1975. The other is the volume of proceedings[11] of the AMS Regional Conference held in Columbia, South Carolina in 1976 edited by S. L. Campbell. It is a new survey book containing 12 papers on the latest applications of generalized inverses. The volume describes changes in research directions and types of generalized inverses since mid-1970s. Prior to this period, due to the applications in statistics, research often centered on generalized inverses for solving linear systems and generalized inverses with least-squares properties. Recent studies focus on such topics as: infinite dimensional theory, numerical computation, matrices of special types (Boolean, integral), matrices over algebraic structures other than real or complex fields, systems theory and non-equation solving generalized inverses.

I have been conducting teaching and research in generalized inverses of matrices since 1976. I gave a course "Generalized Inverses of Matrices" and held many seminars for graduate students majoring in Computational Mathematics in our department. Since 1979, my colleagues and I with graduate students have obtained a number of results on generalized inverses in the areas of perturbation theory, condition numbers, recursive algorithms, finite algorithms, imbedding algorithms, parallel algorithms, generalized inverses of rank-r modified matrices and Hessenberg matrices, extensions of the Cramer rules and the representation and approximation of generalized inverses of linear operators. Dozens of papers are published in refereed journals in China and other countries. They draw attentions from researchers around world. I have received letters from more than ten universities in eight countries, U.S.A., Germany, Sweden, etc. requesting papers or seeking academic contacts. Colleagues in China show strong interests and support in our work, and request systematic presentation of our work. With the support of the Academia Sinica Publishing Foundation and the National Natural Science Foundation of China, Science Press published my book "Generalized Inverses of Matrices and Operators"[169] in Chinese in 1994. That book is noticed and welcomed by researchers and colleagues in China. It has been adopted by several universities as textbook or reference book for graduate students. The book was reprinted in 1998.

In order to improve graduate teaching and international academic exchange, I was encouraged to write this English version based on the Chinese version. This English version is not a direct translation of the Chinese version. In addition to the contents in the Chinese version, this book includes the contents from more than 100 papers since 1994. The final product is an entirely new book, while the spirit of the Chinese version still lives. For example, Sections 2, 3 and 5 of Chapter 3, Section 1 of Chapter 6, Sections 4 and 5 of Chapter 7, Sections 1, 4 and 5 of Chapter 8, Chapters 4, 10 and 11 are all new.

Dr. Wei Yimin of Fudan University in China and Dr. Qiao Sanzheng of McMaster University in Canada were two of my former excellent students. They have made many achievements in the area of generalized inverses and are recognized internationally. I would not possibly finish this book without their collaborations.

We would like to thank Professor A. Ben-Israel, Dr. Miao Jianming of Rutgers University, and Professors R.E. Hartwig, S.L. Campbell and C. D. Meyer, Jr. of North Carolina State University, and Professor C. W. Groetsch of University of Cincinnati. The texts[7], [13] and [65] undoubtedly have had an influence on this book. We also thank Professor Jiang Erxiong of Shanghai University, Professor Cao Zhihao of Fudan University, Professor Wei Musheng and Chen Guoliang of East-China Normal University and Professor Chen Yonglin of Nanjing Normal University for their help and advice in the subject for many years, and my doctoral student Yu Yaoming for typing this book.

I appreciate any comments and corrections from the readers.

Finally, I am indebted to the support by the Graduate Textbook Publishing Foundation of Shanghai Education Committee and Shanghai Normal University.

Wang Guorong
Shanghai Normal University
June 2003

List of Notations

Matrices: For the matrices A and B, and the indices α and β

I	the identity matrix
A^T	the transpose of A
A^*	the conjugate transpose of A
$A^\#$	the weighted conjugate transpose of A
A^{-1}	the inverse of A
$A^{(1)}$	the $\{1\}$-inverse of A
$A^{(1,3)}$, $A^{(1,3M)}$	the $\{1,3\}$-,$\{1,3M\}$-inverses of A
$A^{(1,4)}$, $A^{(1,4N)}$	the $\{1,4\}$-,$\{1,4N\}$-inverses of A
$A^{(1,2)}_{T,S}$	the $\{1,2\}$-inverse of A with prescribed range T and null space S
$A^{(2)}_{T,S}$	the $\{2\}$-inverse of A with prescribed range T and null space S
$A^\dagger$	the Moore-Penrose inverse of A
$A^\dagger_{MN}$	the weighted Moore-Penrose inverse of A
A_d	the Drazin inverse of A
A_g	the group inverse of A
$A_{d,W}$	the W-weighted Drazin inverse of A
$A^{(-1)}_{(L)}$	the Bott-Duffin inverse of A
$A^{(\dagger)}_{(L)}$	the generalized Bott-Duffin inverse of A
$A[\alpha,\ \beta]$ or $A_{\alpha\beta}$	the submatrix of A having row indices α and column indices β

$A[\alpha]$ or A_α	the submatrix $A_{\alpha\alpha}$ of A
$A[\alpha, *]$ or $A_{\alpha *}$	the submatrix of A lying in rows indexed by α
$A[*, \beta]$ or $A_{*\beta}$	the submatrix of A lying in columns indexed by β
$A[\alpha', \beta']$	the submatrix obtained from A by deleting rows indexed by α and columns indexed by β
$A[\alpha']$	the submatrix $A[\alpha', \alpha']$ of A
$\mathrm{adj}(A)$	the adjoint matrix of A
$C_k(A)$	the k-th compound matrix of A
$A(j \to \boldsymbol{b})$	the matrix obtain from A by replacing the j-th column with the vector $\boldsymbol{b}$
$A(\boldsymbol{d}^{\mathrm{T}} \leftarrow i)$	the matrix obtain from A by replacing the i-th row with the row vector $\boldsymbol{d}^{\mathrm{T}}$
$A \otimes B$	the Kronecker product of A and B

Sets and Spaces: For the matrices A and B

$N(A)$	the null space of A
$N_c(A)$	the subspace complementary to $N(A)$
$N(A, B)$	the null space of (A, B)
$R(A)$	the range of A
$R_c(A)$	the subspace complementary to $R(A)$
$R(A, B)$	the range of (A, B)
$\mathbf{R}$, $\mathbb{C}$	the field of real, complex numbers
$\mathbf{R}^n$, $\mathbb{C}^n$	the n-dimensinal real, complex vector space
$\mathbf{R}^{m\times n}$, $\mathbb{C}^{m\times n}$	the set of $m \times n$ matrices over $\mathbf{R}$, $\mathbb{C}$
$\mathbf{R}_r^{m\times n}$, $\mathbb{C}_r^{m\times n}$	the set of $m \times n$ matrices of rank r over $\mathbf{R}$, $\mathbb{C}$

Index sets: For $A \in \mathbf{R}_r^{m\times n}$, and the indices α and β

$Q_{k,n}$	$Q_{k,n} = \{\alpha \ : \ \alpha = (\alpha_1, \cdots, \alpha_k) \ , 1 \le \alpha_1 < \cdots < \alpha_k \le n\}$
$\mathcal{I}(A)$	$\mathcal{I}(A) = \{I \in Q_{r,m} \ : \ \mathrm{rank}(A_{I*}) = r\}$

$\mathcal{J}(A)$	$\mathcal{J}(A) = \{J \in Q_{r,n} \ : \ \text{rank}(A_{*J}) = r\}$
$\mathcal{N}(A)$	$\mathcal{N}(A) = \mathcal{I}(A) \times \mathcal{J}(A) = \{(I,\ J) \in Q_{r,m} \times Q_{r,n} \ : \ \text{rank}(A_{IJ}) = r\}$
$\mathcal{I}(\alpha)$	$\mathcal{I}(\alpha) = \{I \in \mathcal{I}(A) \ : \ \alpha \subset I\}$
$\mathcal{J}(\beta)$	$\mathcal{J}(\beta) = \{J \in \mathcal{J}(A) \ : \ \beta \subset J\}$
$\mathcal{N}(\alpha, \beta)$	$\mathcal{N}(\alpha, \beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta)$

Miscellaneous: For the matrix A

$\det(A)$	the determinant of A
$\frac{\partial}{\partial \lvert A_{\alpha\beta} \rvert} \lvert A \rvert$	the coefficient of $\det(A_{\alpha\beta})$ in the Laplace expansion of $\det(A)$
$\frac{\partial}{\partial a_{ij}} \lvert A \rvert$	the cofactor of a_{ij}
$\text{Vol}(A)$	the volume of A, $\text{Vol}(A) = \sqrt{\sum_{(I,\ J) \in \mathcal{N}(A)} \det^2(A_{IJ})}$
$\text{rank}(A)$	the rank of A
$\text{null}(A)$	the nullity of A
$\text{Ind}(A)$	the index of A
$\text{tr}(A)$	the trace of A
$\lambda(A)$	the spectrum of A
$\sigma(A)$	the set of singular values of A
$\mu_{MN}(A)$	the set of weighted $(M,\ N)$ singular values of A, where M and N are Hermitian positive definite matrices
$\rho(A)$	the spectral radius of A
$\kappa(A)$	the condition number with respect to the inverse of A
$\kappa_{MN}(A)$	the condition number with respect to the weighted Moore-Penrose inverse of A
$\kappa_2(A)$	the condition number with respect to the Moore-Penrose inverse of A
$\kappa_d(A)$	the condition number with respect to the Drazin inverse of A
$\dim(L)$	the dimension of a space L
$P_{L,M}$	the projector on a space L along a space M

P_L	the orthogonal projector on L along $L^\perp$
p.d.	the positive definite
L-p.d.	the L-positive definite
p.s.d.	the positive semi-definite
L-p.s.d.	the L-positive semi-definite
$\|\cdot\|_p$	ℓ_p-norm

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