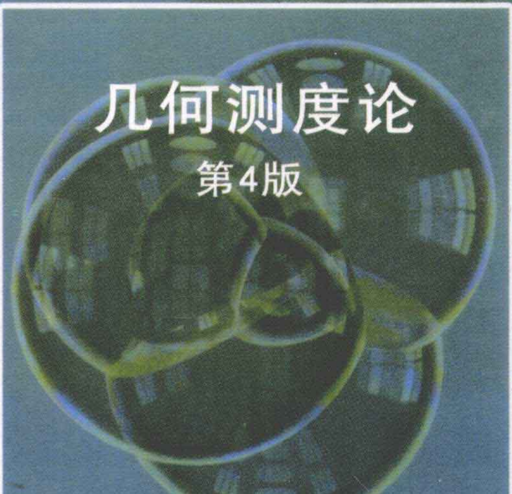
FRANK MORGAN

GEOMETRIC MEASURE THEORY

A BEGINNER'S GUIDE

几何测度论

第4版



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Geometric Measure Theory
A Beginner's Guide

Fourth Edition

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Geometric Measure Theory
A Beginner's Guide

Fourth Edition

Here as a child I watched my mom blow soap bubbles. My dad also encouraged all my interests. This book is dedicated to them with admiration.

Photograph courtesy of the Morgan family; taken by the author's grandfather, Dr. Charles W. Selemeyer

Preface

Singular geometry governs the physical universe: soap bubble clusters meeting along singular curves, black holes, defects in materials, chaotic turbulence, crystal growth. The governing principle is often some kind of energy minimization. Geometric measure theory provides a general framework for understanding such minimal shapes, a priori allowing any imaginable singularity and then proving that only certain kinds of structures occur.

Jean Taylor used new tools of geometric measure theory to derive the singular structure of soap bubble clusters and sea creatures, recorded by J. Plateau over a century ago (see Section 13.9). R. Schoen and S.-T. Yau used minimal surfaces in their original proof of the positive mass conjecture in cosmology, recently extended to a proof of the Riemannian Penrose Conjecture by H. Bray. David Hoffman and his collaborators used modern computer technology to discover some of the first new complete embedded minimal surfaces in a hundred years (Figure 6.1.3), some of which look just like certain polymers. Other mathematicians are now investigating singular dynamics, such as crystal growth. New software computes crystals growing amidst swirling fluids and temperatures, as well as bubbles in equilibrium, as on the front cover of this book. (See Section 16.8.)

In 2000, Hutchings, Morgan, Ritoré, and Ros announced a proof of the Double Bubble Conjecture, which says that the familiar double soap bubble provides the least-area way to enclose and separate two given volumes of air. The planar case was proved by my 1990 Williams College NSF "SMALL" undergraduate research Geometry Group [Foisy *et al.*]. The case of equal volumes in $\mathbf{R}^3$ was proved by Hass, Hutchings, and Schlafly with the help of computers in 1995. The general $\mathbf{R}^3$ proof has now been generalized to $\mathbf{R}^n$ by Reichardt. There are partial results in spheres, tori, and Gauss space, an important example of a manifold with density (see Chapters 18 and 19).

This little book provides the newcomer or graduate student with'an illustrated introduction to geometric measure theory: the basic ideas, terminology, and results. It developed from my one-semester course at MIT for graduate students with a semester of graduate real analysis behind them. I have included a few fundamental arguments and a superficial discussion of the regularity theory, but my goal is merely to introduce the subject and make the standard text. Geometric Measure Theory by H. Federer. more accessible.

Other good references include L. Simon's Lectures on Geometric Measure Theory. E. Guisti's Minimal Surfaces and Functions of Bounded Variation, R. Hardt and Simon's Seminar on Geometric Measure Theory, Simon's Survey Lectures on Minimal Submanifolds, J. C. C. Nitsche's Lectures on Minimal Surfaces (now available in English). R. Osserman's updated Survey of Minimal

Surfaces, H. B. Lawson's Lectures on Minimal Submanifolds, A. T. Fomenko's books on The Plateau Problem, and S. Krantz and H. Parks's Geometric Integration Theory. S. Hildebrandt and A. Tromba offer a beautiful popular gift book for your friends, reviewed by Morgan [14, 15]. J. Brothers and also Sullivan and Morgan assembled lists of open problems. There is an excellent Questions and Answers about Area Minimizing Surfaces and Geometric Measure Theory by F. Almgren [4]. who also wrote a review [5] of the first edition of this book. The easiest starting place may be the Monthly article "What is a Surface?" [Morgan 24].

It was from Fred Almgren, whose geometric perspective this book attempts to capture and share, that I first learned geometric measure theory. I thank many graduate students for their interest and suggestions, especially Benny Cheng, Gary Lawlor, Robert McIntosh, Mohamed Messaoudene, and Marty Ross. I also thank typists Lisa Court, Louis Kevitt, and Marissa Barschdorf. Jim Bredt first illustrated an article of mine as a member of the staff of Link, a one-time MIT student newspaper. I feel very fortunate to have him with me again on this book. I am grateful for help from many friends, notably Tim Murdoch. Yoshi Giga and his students, who prepared the Japanese translation, and especially John M. Sullivan. I would like to thank my new editor, Lauren Schultz, and my original editor and friend Klaus Peters. A final thank you goes to all who contributed to this book at MIT. Rice, Stanford, and Williams. Some support was provided by National Science Foundation grants, by my Cecil and Ida Green Career Development Chair at MIT, and by my Dennis Meenan and Webster Atwell chairs at Williams.

This fourth edition includes updated material and references, recent results on planar soap films (Chapter 13), a new Chapter 18 on Manifolds with Density and Perelman's Proof of the Poincaré Conjecture and a new Chapter 19 on Double Bubbles in Spheres, Gauss Space, and Tori. Gauss space, defined as Euclidean space with Gaussian density, long studied by probabilists, appears along with general manifolds with density in Perelman's original 2003 paper on the Poincaré Conjecture. The proof of the Double Bubble Conjecture in spheres seems inextricably linked to its proof in Gauss space (see Chapter 19).

Bibliographic references are simply by author's name, sometimes with an identifying numeral or section reference in brackets. Following a useful practice of Nitsche [2], the bibliography includes cross-references to each citation.

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Contents

Geometric Measure Theory

1

Geometric measure theory could be described as differential geometry, generalized through measure theory to deal with maps and surfaces that are not necessarily smooth, and applied to the calculus of variations. It dates from the 1960 foundational paper of Herbert Federer and Wendell Fleming on "Normal and Integral Currents," recognized by the 1986 AMS Steele Prize for a paper of fundamental or lasting importance, and earlier and contemporaneous work of L. C. Young [1, 2], E. De Giorgi [1, 3, 4], and E. R. Reifenberg [1–3] (see Figure 1.0.1). This chapter provides a rough outline of the purpose and basic concepts of geometric measure theory. Later chapters take up these topics more carefully.

1.1 Archetypical Problem Given a boundary in $\mathbf{R}^n$, find the surface of least area with that boundary. See Figure 1.1.1. Progress on this problem depends crucially on first finding a good space of surfaces to work in.

1.2 Surfaces as Mappings Classically, one considered only two-dimensional surfaces, defined as mappings of the disc. See Figure 1.2.1. Excellent references include J. C. C. Nitsche's *Lectures on Minimal Surfaces* [2], now available in English, R. Osserman's updated *Survey of Minimal Surfaces*, and H. B. Lawson's *Lectures on Minimal Submanifolds*. It was not until about 1930 that J. Douglas and T. Rado surmounted substantial inherent difficulties to prove that every smooth Jordan curve bounds a disc of least mapping area. Almost no progress was made for higher-dimensional surfaces (until, in a surprising turnaround, B. White [1] showed that for higher-dimensional surfaces the geometric measure theory solution actually solves the mapping problem too).

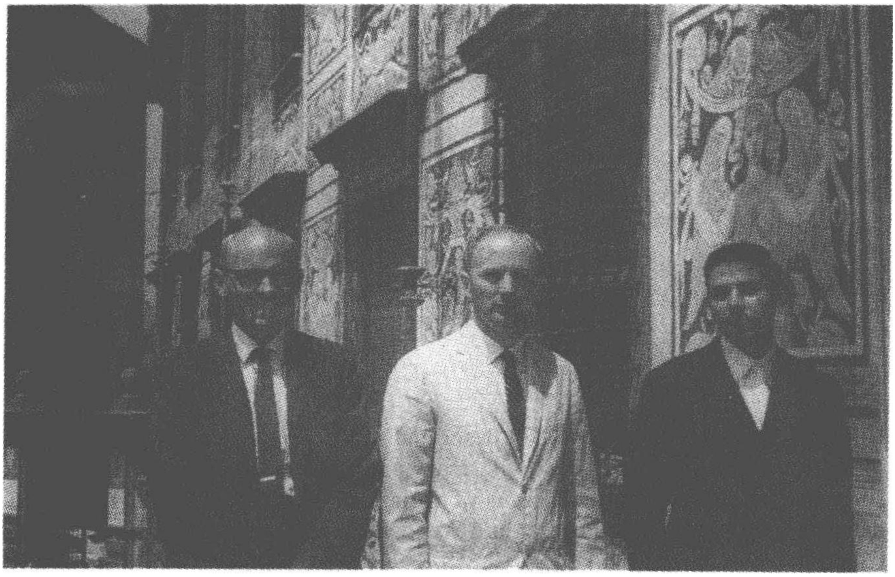

Figure 1.0.1. Wendell Fleming, Fred Almgren, and Ennio De Giorgi, three of the founders of geometric measure theory, at the Scuola Normale Superiore, Pisa, summer, 1965; and Fleming today. Photographs courtesy of Fleming.

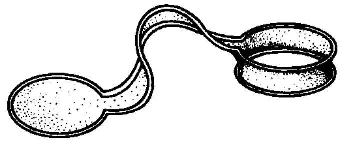

Figure 1.1.1. The surface of least area bounded by two given Jordan curves.

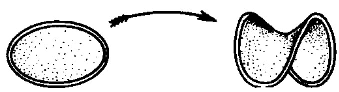

Figure 1.2.1. Surface realized as a mapping, f, of the disc.

Along with its successes and advantages, the definition of a surface as a mapping has certain drawbacks (see Morgan [24]):

1. There is an inevitable *a priori* restriction on the types of singularities that can occur;
2. There is an *a priori* restriction on the topological complexity; and
3. The natural topology lacks compactness properties.

The importance of compactness properties appears in the direct method described in the next section.

1.3 The Direct Method

The direct method for finding a surface of least area with a given boundary has three steps.

1. Take a sequence of surfaces with areas decreasing to the infimum.
2. Extract a convergent subsequence.
3. Show that the limit surface is the desired surface of least area.

Figures 1.3.1–1.3.4 show how this method breaks down for lack of compactness in the space of surfaces as mappings, even when the given boundary is the unit

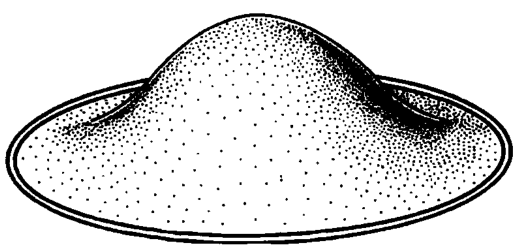

Figure 1.3.1. A surface with area $\pi + 1$.

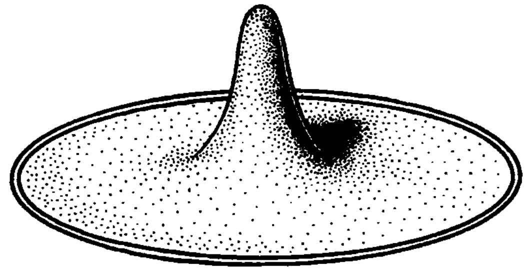

Figure 1.3.2. A surface with area $\pi + \frac{1}{4}$.

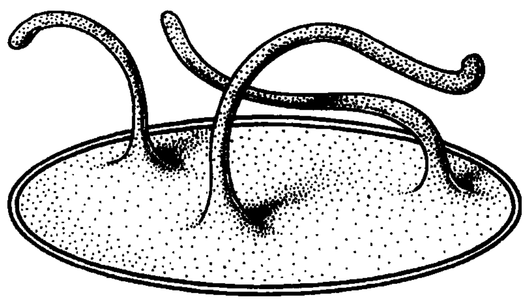

Figure 1.3.3. A surface with area $\pi + \frac{1}{16}$.

circle. By sending out thin tentacles toward every rational point, the sequence could include all of $\mathbf{R}^3$ in its closure!

1.4 Rectifiable Currents An alternative to surfaces as mappings is provided by *rectifiable currents*, the m-dimensional, oriented surfaces of geometric measure theory. The relevant functions $f: \mathbf{R}^m \to \mathbf{R}^n$ need not be smooth but merely *Lipschitz*; that is,

$$|f(x) - f(y)| \leq C|x - y|,$$

for some "Lipschitz constant" C.

Fortunately, there is a good m-dimensional measure on $\mathbf{R}^n$, called *Hausdorff measure*, $\mathscr{H}^m$. Hausdorff measure agrees with the classical mapping area of an embedded manifold, but it is defined for all subsets of $\mathbf{R}^n$.

A Borel subset B of $\mathbf{R}^n$ is called $(\mathscr{H}^m, m)$ *rectifiable* if B is a countable union of Lipschitz images of bounded subsets of $\mathbf{R}^m$, with $\mathscr{H}^m(B) < \infty$. (As usual,

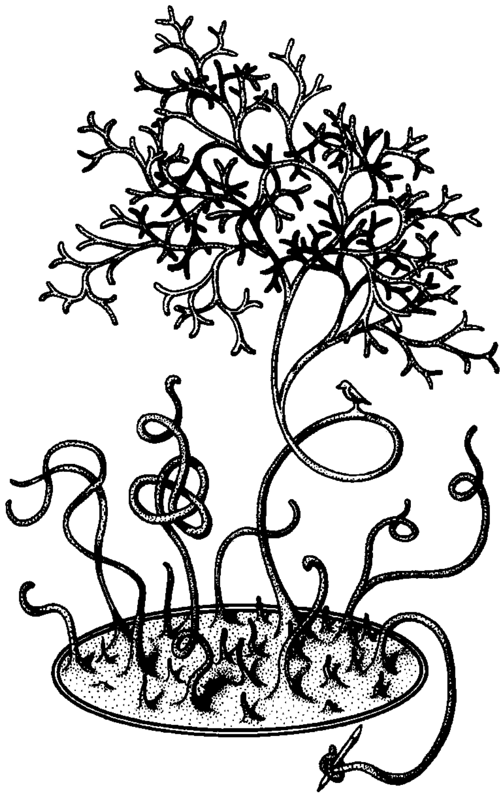

Figure 1.3.4. A surface with area $\pi + \frac{1}{64}$.

we will ignore sets of $\mathcal{H}^m$ measure 0.) That definition sounds rather general, and it includes just about any "m-dimensional surface" I can imagine. Nevertheless, these sets will support a kind of differential geometry: for example, it turns out that a rectifiable set B has a canonical tangent plane at almost every point.

Finally, a *rectifiable current* is an oriented rectifiable set with integer multiplicities, finite area, and compact support. By general measure theory, one can integrate a smooth differential form φ over such an oriented rectifiable set S, and hence view S as a *current*; that is, a linear functional on differential forms,

$$\varphi \mapsto \int_S \varphi.$$

This perspective yields a new natural topology on the space of surfaces, dual to an appropriate topology on differential forms. This topology has useful compactness properties, given by the fundamental compactness theorem in Section 1.5. Viewing rectifiable sets as currents also provides a boundary operator ∂ from m-dimensional rectifiable currents to $(m-1)$-dimensional currents, defined by

$$(\partial S)(\varphi) = S(d\varphi),$$

where $d\varphi$ is the exterior derivative of φ. By Stokes's theorem, this definition coincides with the usual notion of boundary for smooth, compact, oriented manifolds with boundary. In general, the current ∂S is not rectifiable, even if S is rectifiable.

1.5 The Compactness Theorem *Let c be a positive constant. Then the set of all m-dimensional rectifiable currents T in a fixed large closed ball in* $\mathbf{R}^n$, *such that the boundary* ∂T *is also rectifiable and such that the area of both T and* ∂T *is bounded by c, is compact in an appropriate weak topology.*

1.6 Advantages of Rectifiable Currents Notice that rectifiable currents have none of the three drawbacks mentioned in Section 1.2. There is certainly no restriction on singularities or topological complexity. Moreover, the compactness theorem provides the ideal compactness properties. In fact, the direct method described in Section 1.3 succeeds in the context of rectifiable currents. In the figures of Section 1.3, the amount of area in the tentacles goes to 0. Therefore, they disappear in the limit in the new topology. What remains is the disc, the desired solution.

All of these results hold in all dimensions and codimensions.

1.7 The Regularity of Area-Minimizing Rectifiable Currents One serious suspicion hangs over this new space of surfaces: The solutions they provide to the problem of least area, the so-called area-minimizing rectifiable currents, may be generalized objects without any geometric significance. The following interior regularity results allay such concerns. (We give more precise statements in Chapter 8.)

1. A two-dimensional area-minimizing rectifiable current in $\mathbf{R}^3$ is a smooth embedded manifold.
2. For $m \leq 6$, an m-dimensional area-minimizing rectifiable current in $\mathbf{R}^{m+1}$ is a smooth embedded manifold.

Thus, in low dimensions the area-minimizing hypersurfaces provided by geometric measure theory actually turn out to be smooth embedded manifolds. However, in higher dimensions, singularities occur, for geometric and not merely technical reasons (see Section 10.7). Despite marked progress, understanding such singularities remains a tremendous challenge.

1.8 More General Ambient Spaces Basic geometric measure theory extends from $\mathbf{R}^n$ to Riemannian manifolds via C^1 embeddings in $\mathbf{R}^n$ or Lipschitz charts. Ambrosio and Kirchheim among others have been developing an intrinsic approach to geometric measure theory in certain metric spaces.