# Contributions to Nonlinear Functional Analysis

Edited by Eduardo H. Zarantonello

Proceedings of a Symposium
Conducted by the Mathematics Research Center,
The University of Wisconsin, Madison
April 12–14, 1971

Academic Press
New York · London 1971

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

*United Kingdom Edition published by*
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DD

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 70-182611

PRINTED IN THE UNITED STATES OF AMERICA

# Foreword

This volume contains the complete texts of the fifteen addresses to the Symposium on Nonlinear Functional Analysis, held in Madison on April 12-14, 1971, under the sponsorship of the Mathematics Research Center, University of Wisconsin. There were six sessions covering the following areas: I. Topological degree and bifurcation, II. Monotonicity, III. Convexity, IV. Integral equations, V. Evolution equations, and VI. Partial differential equations. The sessions were chaired by:

Professor Erich H. Rothe, University of Michigan
Professor Eduardo H. Zarantonello, Mathematics Research Center, University of Wisconsin
Professor Victor Klee, University of Washington
Professor John Nohel, University of Wisconsin
Professor Jürgen K. Moser, New York University
Professor Charles C. Conley, University of Wisconsin

The program committee consisted of Professors P. H. Rabinowitz, R. E. L. Turner, and L. Rall, with the editor as the chairman. Mrs. Gladys Moran was the symposium secretary, and it is to her experience, intelligent dedication, and inexhaustable enthusiasm that this conference owed its perfect organization. The preparation of the manuscripts for publication was in the able hands of Mrs. Dorothy Bowar. To them both I wish to extend my appreciation for their invaluable assistance.

Eduardo H. Zarantonello

# Preface

Linearity is such a deep-seated notion among mathematicians that any outside venture, especially in functional analysis, is immediately qualified as "nonlinear," as if linearity were the normal way of life in mathematics. Such ventures beyond the linear are seldom strictly nonlinear, for they also apply to linear situations, and more often than not it is there where they are at their best, if only a banal best. Strictly speaking, it is only in a context in which linearity makes sense that one can speak of nonlinearity. As the outlying field of nonlinearity is being explored and developed, it is becoming clear that linearity is just one of many parcels of mathematical territory, the first to be settled and at that a thin and narrow one, and that the tribute paid to it is no longer unquestionable. Hopefully, the term nonlinear will disappear from analysis, to be replaced by a host of new names making for a more precise and representative nomenclature, and one can foresee the time—not so far off—when linearity rather than its absence will have to be qualified.

Two of the main ideas in the contemporary scene of functional analysis—as distinguished from linear functional analysis—are topological degree and monotonicity. Topological degree, in the form of the homotopy invariance of the topological index, has been a prime source of existence proofs since its formulation by Schauder and Leray in 1934. It is also an important instrument in bifurcation theory. However, the requirement in existence questions that the operators involved be compact considerably restricts its domain of application. This gap was partially filled by monotone operators which need be neither compact nor continuous, and which originated in 1960 out of the need to have something in higher dimensions corresponding to increasing functions on the real line. Monotonicity belongs to the lineage of ideas started by Picard's successive approximations and later represented by the Banach contraction principle, but it goes a good deal beyond. The theory of convex functionals and duality, now being vigorously pursued, falls partly within its realm through the fact that subgradients of convex functionals are monotone mappings. Of course, many other techniques are used in functional analysis. Among these one should mention traditional "hard" analysis, whose role in bringing specific problems into the fold of general ideas is permanently assured in this field. Indeed, it is by means of its sophisticated techniques that such essentials as *a priori* bounds, estimates, coerciveness, contractiveness, and monotonicity are established.

The papers presented in this volume strongly reflect the above-mentioned tendencies in functional analysis. We shall briefly categorize them within such context: L. Nirenberg presents an extension of Leray-Schauder degree and gives an application to a nonlinear elliptic boundary value problem. P. H. Rabinowitz applies degree theory to prove the existence of global continua of solutions of nonlinear eigenvalue problems. Further results about continua of solutions are obtained by R. E. L. Turner using the notion of transversality. K. Kirchgässner shows how variational structure can be used to study some local questions in bifurcation theory.

A large number of papers touch on the notion of monotone operators: H. Brezis presents a brief survey of monotonicity theory, discusses the maximality of the sum of maximal monotone operators, and gives applications to partial differential equations. M. G. Crandall offers a nonlinear version of the Hille-Yosida theorem. Integral equations of the Hammerstein and Urysohn type are the subject of F. E. Browder's article. J. L. Lions gives an extension of boundary layer theory to variational inequalities of elliptic, parabolic, and hyperbolic type. A version of the penalty method for the Navier-Stokes equations is presented by H. Fujita. Three communications deal with convexity: J. J. Moreau discusses various types of weak solutions for minimizing problems in the spirit of duality theory for convex functionals. The duals of convex integral functionals constructed out of one-parameter families of convex functionals are studied in the article by R. T. Rockafellar. E. H. Zarantonello takes up the study of projections on convex sets in Hilbert space, and develops a spectral theory for a class of operators not necessarily linear, extending the classical one for self-adjoint linear operators.

Analysis in its more classical form is represented by three papers: By use of the maximum principle, J. Serrin obtains *a priori* estimates for gradients of solutions of partial differential equations of parabolic and elliptic type. P. D. Lax discusses recent developments in conservation laws, and J. J. Levin and D. F. Shea investigate the asymptotic behavior of the solutions of certain nonlinear integral equations of Volterra type.

Eduardo H. Zarantonello

## Contents

Numbers in parentheses refer to AMS (MOS) 1970 subject classifications.

CONTENTS

# Generalized Degree and Nonlinear Problems

*L. NIRENBERG*

§1. In this talk, which is based on [6], we will illustrate the use of some topological techniques in solving nonlinear problems. To start with a simple and well known example, let $T$ be a continuous mapping of the closed unit ball $B$ in $R^d$ into $R^{d^*}$ - we wish to solve the equation

$$T(x) = 0 .$$

The topological techniques yield conditions on the boundary values $T_0$ of $T$ which ensure that for <u>every</u> extension $T$ of $T_0$ inside $B$ the equation $T(x) = 0$ is always solvable. Assume that $T_0(x) \neq 0$ on $\partial B$, then one has the following elementary but basic result expressed in terms of the normalized map $\psi(x) = T_0(x)/|T_0(x)|$ mapping $\partial B = S^{d-1}$ into $S^{d^*-1}$:

<u>Proposition.</u> A necessary and sufficient condition that for every extension $T$ of $T_0$ the equation $T(x) = 0$ is always solvable is that the homotopy class of $\psi$ be nontrivial

This theorem yields useful results only in case $d^* \leq d$. If $d^* = d$ the homotopy class of $\psi$ being nontrivial means that the degree of the map $\psi$, i.e., the number of times the image sphere is covered (counted algebraically), is different from zero. This number $\nu$ is also equal to the degree of the map $T$ at the origin in the image space, i.e. the number of times the origin is covered (counted algebraically).

Consider now an infinite dimensional Banach space X and a continuous map T of B , the closed unit ball in X, into X , with I - T = a compact operator K . The Leray-Schauder theory, which has been one of the most useful techniques in attacking nonlinear problems, is a generalization of the preceding remarks to this situation (B may be the closure of any open set in X ). If $T_0(x) \neq 0$ , where $T_0 = T|_{\partial B}$ , then the mapping T has, again, an integral valued degree $\nu$ at the origin; if $\nu \neq 0$ then $T(x) = 0$ is solvable in B . The degree $\nu$ depends only on $T_0$ , in fact only on the homotopy class of $T_0$ within the class of operators such that $I - T_0$ is compact and $T_0(x) \neq 0$ on $\partial B$.

If the range of T is contained within a linear subspace Y of X , $Y \neq X$ , then the degree $\nu$ of T at the origin is necessarily zero - since it is the same for all points in a neighbourhood of the origin and, at a point off Y , and thus not in the range of T , it vanishes. In this lecture we shall describe an extension of the Leray-Schauder theorem to such a situation and an application to a nonlinear elliptic boundary value problem. In recent years extensions of the Leray-Schauder theory have been made in various directions. The result presented here (and in [6]) is part of a more general development (see [7], [3], [1], [2]).

Consider a mapping $T : B \to Y \subset X$ as above, with $I - T = K$ compact, $T(x) \neq 0$ on $\partial B$ , and Y a closed subspace having finite codimension i. We wish to present a condition on $T_0 = T|_{\partial B}$ to ensure that the equation $T(x) = 0$ is solvable in B for <u>any</u> extension T of $T_0$ inside B - of the form I - compact, and having range in Y . Maps $T_0$ of $\partial B$ into Y with this property are called "essential". Whether $T_0$ is essential or not depends only on its homotopy class (always of the form I-compact) of maps into $Y^* = Y \setminus \{0\}$. For Y = X this is proved in Granas [4], Theorem 22, and the proof is easily extended for any subspace Y . For T = I - K the compact operator K may be approximated by one with finite dimensional range and hence, as one easily sees, the operator $T_0$ may be deformed within its homotopy class to an operator of the form $I - K_1$ , mapping

mapping $\partial B$ into $Y$, with $K_1$ mapping into a finite dimensional space. Thus if we write $X$ as a directed sum

$$X = Y \oplus Z , \quad \dim Z = i ,$$

so that any vector $x$ in $X$ has the unique decomposition $x = y + z$, with $y \varepsilon Y$, $z \varepsilon Z$, we may suppose that $T_0$ has the form

$$T_0(x) = T_0(y+z) = y + z - K_1(x) = y - K_2(x) ,$$

where $K_2$ is a map of $\partial B$ into a finite dimensional subspace $V$ of $Y$. Decomposing $Y$ as a direct sum

$$Y = W_1 \oplus V ,$$

with $W_1$ a closed linear subspace of $Y$, so that any $x \varepsilon X$ now has the unique decomposition $x = y + z = w_1 + v + z$, $w_1 \varepsilon W_1$, $v \varepsilon V$, $z \varepsilon Z$, we have

$$T_0(x) = w_1 + v - K_2(x) = w_1 - K_3(x)$$

where the range of $K_3$ is in $V$.

Since $T_0(x) \neq 0$ for $x \varepsilon \partial B$ we see that $K_3(v+z) \neq 0$ for $v + z \varepsilon \partial B$. Hence we may deform $T_0$ via the deformation

$$T_{0t}(x) = T_{0t}(w_1+v+z) = w_1 - K_3(tw_1+v+z), \quad 0 \leq t \leq 1$$

to the map

$$T_{01}(x) = w_1 - K_3(v+z)$$

lying in the same homotopy class. We may therefore suppose that $T_0$ has this very special form, namely, with $V \oplus Z = W$ so that $x = w_1 + w$, we may suppose that

$$T_0(x) = T_0(w_1 + w) = w_1 + \Phi(w)$$

where $\Phi$ is a continuous map of the closed unit ball in W into the linear subspace V of W. We shall express the condition for $T_0$ to be "essential" in terms of the map $\Phi$ which does not vanish for $\|w\| = 1$. Suppose dim W = d, dim V = $d^*$, $d - d^* = i$; set

(1) $$\Psi(w) = \frac{\Phi(w)}{\|\Phi(w)\|} \quad \text{for} \quad \|w\| = 1 .$$

Then we may consider $\Psi$ as a mapping of the sphere $S^{d-1}$ to $S^{d^*-1}$.

<u>Theorem l.</u> $T_0$ is "essential" if and only if the map $\Psi$ has nontrivial stable homotopy (defined by suspension).

A proof is given in [6]; in proving sufficiency one first approximates I - T by an operator mapping into a finite dimensional space - reducing the problem to that for finite dimensional X. In this case one then applies the Proposition above by showing that the homotopy class of $\frac{T_0(x)}{|T_0(x)|}$, mapping the unit sphere in X into that in Y, is obtained from the mapping $\Psi$ by repeated suspensions.

§2. The application that we present grew out of a result of Landesman and Lazer [5] and we shall first describe their result in a slightly restricted form. It concerns a nonlinear elliptic boundary value problem for a real function u in a bounded domain $\mathfrak{D} \subset R^n$ with smooth boundary $\Gamma$. (All functions, coefficient of equations, etc., are assumed to be real and smooth in $\bar{\mathfrak{D}}$.) Let L be a linear formally self-adjoint elliptic second order operator in $\bar{\mathfrak{D}}$ and consider the problem

(2) $$Lu = f(x) - g(u) \quad \text{in} \quad \mathfrak{D}, \quad u = 0 \quad \text{on} \quad \Gamma ,$$

with f a given (smooth) function; g(u) is continuous and has limits

$$\lim_{u \to \pm\infty} g(u) = g(\pm\infty)$$

with

(3) $$g(-\infty) < g(u) < g(\infty) .$$

Assume that $\ker L$ , i.e. the space of solutions of

(4) $$Lu = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma ,$$

is one dimensional - spanned by the function $w$ . Then from (3) one easily derives a necessary condition for solvability of (2); taking $L_2$ scalar product ( , ) of (2) with $w$ we find

$$(f - g, w) = (Lu, w) = (u, Lw) = 0$$

and using the bounds (3) we obtain the necessary condition

(5) $$g(-\infty) \int_{w>0} w \, dx + g(\infty) \int_{w<0} w \, dx < (f, w) < g(\infty) \int_{w>0} w \, dx + g(-\infty) \int_{w<0} dx .$$

The surprising result of [5] is that (5) is also sufficient for solvability of (2).

We shall present a generalization of this result, based on Theorem 1, concerning elliptic systems of $N$ equations for $N$ functions $u = (u^1, \ldots, u^N)$ in $\Omega$. Let $L$ be a linear elliptic system of order $m$ , and consider vector functions $u$ satisfying homogeneous boundary conditions $Bu = 0$ which are "nice" relative to $L$ , i.e., so called, coercive boundary conditions. We will not describe these in any detail except to say that $\ker L$ = the space of functions $u$ satisfying $Lu = 0$, and $Bu = 0$, on $\Gamma$ is finite dimensional, spanned, say, by the (vector) functions $w_1, \ldots, w_d$; furthermore, the range of $L$ (acting on smooth functions satisfying $Bu = 0$) consists of the smooth functions which are

$L_2$-orthogonal to a finite number of smooth functions $w'_1, \ldots, w'_{d^*}$. The elliptic operator has an index

$$i = \text{ind } L = d - d^* ,$$

and we shall assume that $i = d - d^* \geq 0$.

We shall also make the following hypothesis concerning $\ker L$, the space of functions spanned by $w_1, \ldots, w_d$:

(UC) $w = 0$ <u>is the only function in</u> $\ker L$ <u>which vanishes on a set of positive measure in</u> $\mathcal{D}$.

The nonlinear system to be solved is of the form

$$(6) \qquad L u = g(x, D^\alpha u) \text{ in } \mathcal{D}, \quad Bu = 0 \text{ on } \Gamma ,$$

where $g$ is a smooth bounded $N$ vector for $x \in \bar{\mathcal{D}}$ and all values of the other arguments; $g$ depends on $u$ and its derivatives $D^\alpha u$ up to order $m-1$. For all arguments $\eta = \{\eta^\alpha\} \neq 0$ with $|\alpha| \leq m-1$ (symmetric in the indices $\alpha_i$ of $(\alpha = \alpha_1 \ldots \alpha_n)$) we suppose that

$$(7) \qquad h(x, \eta) = \lim_{r \to \infty} g(x, r\eta)$$

and that the convergence is uniform on $\bar{\mathcal{D}} \times \{|\eta| = 1\}$. We shall give sufficient conditions on $h$ to ensure the solvability of (6).

For $a \in S^{d-1}$ define the map $\phi : S^{d-1} \to R^{d^*}$ by

$$\phi_\beta(a) = (h(x, D^\alpha \sum a_j w_j(x)), w'_\beta) , \quad \beta = 1, \ldots, d^* .$$

As a consequence of the hypothesis (UC) one may prove (as in [6] with the aid of Lemmas 1 and 2 there) that the mapping $\phi$ is continuous. Assume that $\phi(a) \neq 0$ for $a \in S^{d-1}$ and set

$$\psi(a) = \frac{\phi(a)}{|\phi(a)|} , \quad \psi : S^{d-1} \to S^{d^*-1} .$$

Theorem 2. If $\psi$ has nontrivial stable homotopy then (6) is solvable.

By a solution we mean a function in $C^{m-1}$ with derivatives of order $m$ in $L_p$ for large $p$. If $g$ is smooth then using well known regularity theory, it follows that any such solution is smooth.

The proof of the theorem is the same as that of Theorem 2 in [6].

Remarks. (i) If $d = d^*$ then "$\psi$ has nontrivial stable homotopy" means simply that $\psi$ is homotopically nontrivial, i.e. has nonzero degree. In this case one proves the result using the Leray-Schauder degree.

In case $N = 1$, $d = d^* = 1$, and $g = g(x,u)$ depends only on $u$ and not on its derivatives, then $h(x,\eta)$ corresponds to

$$h_{\pm}(x) = h(x, \pm 1) = \lim_{u \to \pm\infty} g(x,u) .$$

In this case the condition that $\psi$ be homotopically nontrivial means that

$$A_1 = \int_{w>0} h_+ w' \, dx + \int_{w<0} h_- w' \, dx$$

and

$$A_2 = \int_{w<0} h_+ w' \, dx + \int_{w>0} h_- w' \, dx$$

have opposite signs. Theorem 2 then contains the result of Landesman and Lazer described above as a special case.

(ii) In the theorem, $\mathfrak{D}$ may be a manifold, and the system of vectors $u(x)$ may be replaced by cross sections of a vector bundle in which $L$ acts; $g$ is then also required to take its values there. The vector bundle is supposed to have a Hermitian metric, and the maps $\phi$ and $\psi$ may be defined as before. Their definitions depend on choice of bases $w_i$ and $w'_\alpha$ and so are not canonical. However the condition on the stable homotopy of $\psi$ is independent of these choices.

(iii) Since it is not known how to determine whether a map $\psi$ has nontrivial stable homotopy, the theorem is not readily applicable.

## REFERENCES

1. K. D. Elworthy. Some problems in algebraic topology. Fredholm maps and $GL_C(E)$ structures. Ph. D. Thesis, Oxford Univ. 1967.

2. K. D. Elworthy, A. Tromba. Differential structures and Fredholm maps on Banach manifolds. Global Analysis. Proc. Symp. Pure Math. 15 Amer. Math. Soc. 1970, p. 45-94.

3. K. Geba. Algebraic topology methods in the theory of compact fields in Banach spaces. Fund. Math. 54 (1964), p. 177-209.

4. A. Granas. The theory of compact vector fields and some of its applications to topology of functional spaces. I. Rozprawy Mat. 30 (1962) .

5. E. M. Landesman and A. C. Lazer. Nonlinear perturbations of linear elliptic boundary value problems at resonance. J. Math. Mech. 19 (1970), p. 609-623.

6. L. Nirenberg. An application of generalized degree theory to a class of nonlinear problems. Proc. Symp. Functional Analsis, Liège, Sept. 1970; to appear.

7. A. S. Švarc. The homotopic topology of Banach spaces. spaces. Dokl. Akad. Nauk SSSR 154 (1964), p.61-63; Soviet Math. Dokl. 5 (1964), p. 57-59.

The work for this paper was supported by the U. S. Air Force Contract AF-49 (638)-1719.



Courant Institute of Mathematical Sciences
New York University
New York, New York

Received April 12, 1971