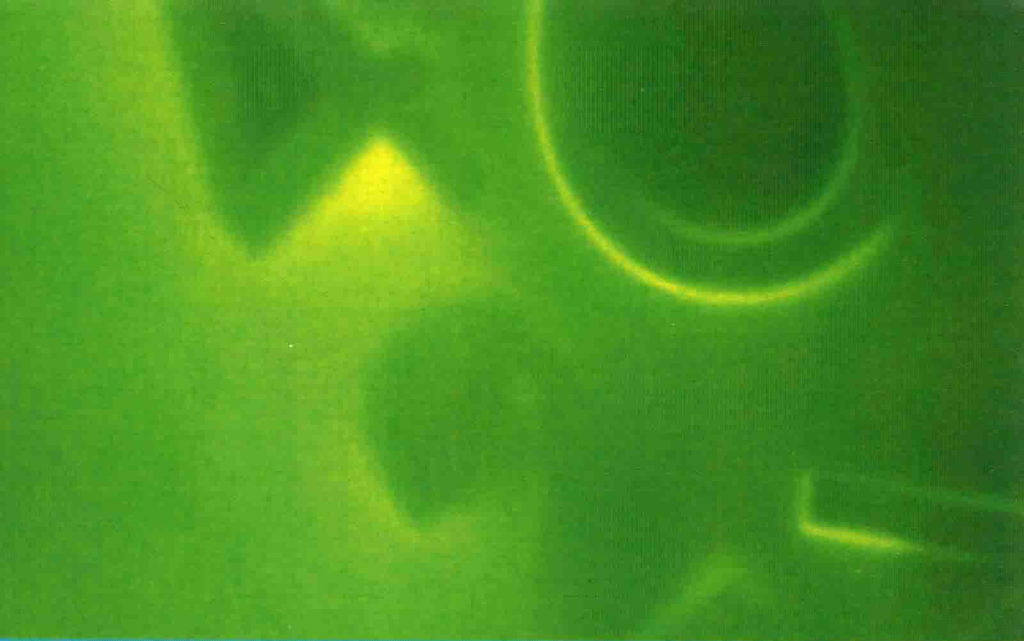

Surajit Borkotokey

Advanced Topics in Fuzzy Algebra

Recent Developments

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Preface:

Since its inception in 1965, by Lotfi Zadeh, fuzzy set theory has become instrumental in application to almost all branches of natural and social sciences. Every discipline, having been contemplated with crisp mathematical formulations has now developed *a* parallel notion in fuzzy environment as well. In 1971, Azriel Rosenfeld introduced the notion of fuzzy algebra and since then a large number of research works has been carried out in various topics of Algebraic structures. Even though Fuzzy Algebra has been studied for over thirty years, number of books on fuzzy algebra is seemingly less. It is worth mentioning that John N Mordeson and D S Malik in their book "Fuzzy Commutative Algebra" have made an exhaustive deliberation of the subject. Their effort in compiling the contemporary works in a lucid manner needs appreciation by every one working in the field. In this book, some advanced topics of abstract algebra in fuzzy environment have been discussed. This is the first book in fuzzy algebra which covers study of various fuzzy algebraic structures incorporating the notion of a Γ-set in it, though there are several papers on these concepts. Another very important notion we have discussed here is that of tensor products between fuzzy R-submodules. The enlightened reader may find it interesting to see that this notion comes from its crisp counterpart in a very natural way.

To the delight of the fuzzy algebraic topologists, we have also tried to bring about a relationship between *a* fuzzy topological vector space and the fuzzy topological space consisting of all fuzzy subspaces of a given vector space.

The intended readers of this book are senior undergraduate students, graduate students, researchers who deal with subjects of fuzzy algebra and fuzzy algebraic topology. In order to master all the materials discussed here, the readers would probably be required to have some background of algebraic structures and topology.

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Dibrugarh, May 2010 Surajit Borkotokey

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Advanced Topics in Fuzzy Algebra

1. Introduction

1.1 Introduction and Historical Developments

" *The theory of fuzzy sets is basically a graded concept – a theory where everything is a matter of degree or , to put it in other way – everything has elasticity.* "

-H.J. Zimmermann

In practice we seldom get the certainty of an event, or the precise-ness of *a* statement. Similarly, it is often difficult to model real life situations by the precise mathematical objects. In practice, we frequently encounter ill defined mathematical concepts because of our inability to relate their abstractness with human cognition. For example : " The class of tall people in the class", "the class of real numbers which are very close to 2", and so on. This impreciseness of objects was first dealt with mathematical formulations by L.A.Zadeh [64] in 1965 in his celebrated paper entitled " Fuzzy Sets", however some ideas presented in the paper were envisioned some thirty years before by the American philosopher Max Black [7], who wrote : It is a paradox , whose importance familiarity fails to diminish, that the most highly developed and useful scientific theories are ostensibly expressed in terms of objects never encountered in experience. The line traced by a draftsman, no matter how accurate, is seen beneath the microscope as a kind of corrugated trench, far removed from the ideal line of pure geometry. And the 'point planet' of Astronomy , the 'Perfect Gas' of Thermodynamics or the 'Pure Species' of Genetics are equally remote from realization. To say that all language (Symbolism or Thought) is vague, is a favorite method for evading the problems involved and lack of analysis has the disadvantage of tempting even the most eminent thinkers into the appearance of absurdity. We shall not assume that the "laws" of Logic or Mathematics prescribe modes of existence to which intelligible discourse must necessarily conform. It is argued, on the contrary, that the deviations from the Logical or Mathematical standards of precision are all pervasive in symbolism: that to label them as subjective observations sets an impassable gulf between formal laws and experiences and leaves the usefulness of the formal sciences and insoluble mystery.

The building block of Mathematics is the theory of Sets. If the vagueness is considered in Astronomy , Thermodynamics or in Genetics as rightly described by Black, the same ill-definedness can be thought of in the case of the theory of sets. Eventually there developed the theory of Fuzzy Sets. A fuzzy set , as defined by Zadeh is characterized by a membership function which assigns to each object a grade of membership ranging from 0 to 1.

We can term this shift from the crisp sense to fuzzy sense as a paradigm shift. According to Klir and Yuan[35], a paradigm shift is initiated by emerging problems that are difficult or impossible to deal with , by the current and available paradigms. Since the new paradigm is initially not well developed the position of its proponents is weak, however it gains momentum when people start working with it.

Therefore whatever Zadeh introduced has become a paradigm shift and after a period of four decades since its inception , it effectively applies to theory of Economics, Psychology, Artificial Intelligence , Computer Sciences, Control System, Expert Management System , Artificial Neural network System, Graph Theory, Operation Research, Image Processing and Pattern Recognition ,Robotics, Bio-Mathematics , Topology , Measure Theory, Operator Theory, Algebra and many more, thus gaining its momentum rapidly through almost all branches of pure and applied sciences. Thousands of research papers in various international , national, and regional journals have been published till date on every such branch.

The concept of Fuzzy Topology comes almost naturally from the concept of fuzzy sets which is merely a generalization of the notion of ordinary topology with the inclusion of one or two extra conditions. The theory flourished to make a great deal of literature in fuzzy topological spaces. The names behind these developments include mathematicians like Chang[15], Lowen[45], Katsaras[31,32,33] , Foster [21] and many more.

In 1971, Rosenfeld [53] introduced the notion of a fuzzy subgroup. And that was the land mark of its own kind in the study of fuzzy-ness over algebraic structures, which indeed paved the way for a massive influx of new ideas as well as new theories in the literature. Rosenfeld was the first in this direction to apply the very idea of fuzzy-ness into the elementary theory of groups and groupoids.

Despite the fact that the development of Fuzzy Algebra is more recent than that of some other branches of Mathematics, all such activities and investigations on the field through different approaches have created an extensive literature. It is difficult to make an account of all these developments here. Therefore here we shall confine and concentrate on only those topics which are relevant to the purpose of our book.

The idea of Γ-semigroup was initiated by Sen and Saha [57] in 1986 , however they introduced this concept on the basis of exhaustive earlier works on Γ -Ring by several workers , viz [43,51,52]. The ring of all square matrices over a division ring plays a vital role in classical ring theory. However when one considers the set of all rectangular matrices (of the same type) , there appears to be no natural way of introducing a binary ring multiplication into it .Various authors like Nobusawa [51] ,Lister [43] , have tried to overcome this difficulty by considering a natural ternary multiplication instead of a binary one. This led them to introduce the notion of a Γ-Ring. In 1995 Y.B. Jun [28] introduced the notion of fuzzy ideals as well as that of a prime ideal in a Γ -Ring extending the analogous definition for an ordinary ring , given by Liu [44] , Mukherjee & Sen [48] and Swami and Swami [60]. In his paper Jun [28] defined the product of a fuzzy left ideal , an additive abelian group Γ and a fuzzy right ideal which is contained in their intersection. The idea of fuzzy prime ideals in a Γ-ring came naturally from the notion of this product . In classical ring theory , the image set of a fuzzy prime ideal as defined by Yue [63] , (further generalized by Mukherjee and Sen [48]) consists of only two elements . In a Γ-Ring also the same situation prevails which was rightly shown in Jun's paper.

In 2001, Yunji Zhang [67] introduced the concept of homologous fuzzy subgroups of a group and investigated the properties of fuzzy centralizers, fuzzy normalizers and fuzzy abelian groups in the light of homologous-ness. He has further verified that 'homologous' is an equivalence relation in the set of all fuzzy subgroups of a group. According to the homologous relation, the set of all fuzzy subgroups of a group can be classified and each such class be called 'the class of homologous fuzzy subgroups'.

Jun and Hong [25] in their paper in 1995 , gave the definition of normalized fuzzy ideals and fuzzy maximal ideals in Γ-rings. They have shown that fuzzy maximal ideals are normalized and take only values 0 and 1; and thus gave a characterization of normalized fuzzy ideals and studied few related properties. Their intension was to correlate the concepts of maximality of a fuzzy ideal and the corresponding concept of an ordinary ideal in a Γ-Ring.

Nobuaki Kuroki [39] in his paper introduced the notion of a fuzzy congruence on groupoid and group and characterized it in terms of fuzzy normal subgroups. He further developed an equivalence between the group structures in fuzzy sense in terms of composition among the fuzzy subgroups and their commutativity under the same composition.

Concept of fuzzy relations on arbitrary sets was initiated by Zadeh [65] himself. Later the concept had been extended to rings and groups by Mordeson and Malik [46] in 1991. They investigated various properties of fuzzy right and left ideals of the ring formed by the Cartesian product of two other rings in an extensive way. They gave one very interesting result viz. " For the Cartesian product to be a fuzzy ideal , both the constituent fuzzy subsets need not be fuzzy ideals however the converse follows naturally". With an example they have shown that the weaker fuzzy subset of a ring induced by a fuzzy relation need not be a fuzzy ideal of the ring. Earlier Bhattacharya and Mukherjee [6] also studied fuzzy relations on groups. Malik et al. [46] in their paper contradicted few of their earlier results.

In 1988,, a new approach to the concept of Cartesian product , relations and functions in fuzzy set theory was established by K.A.Dib and N.L.Youssef [18]. They introduced the notion of fuzzy Cartesian product using a suitable lattice. A fuzzy relation is then defined to be a subset of the fuzzy Cartesian product , analogous to the definition given in crisp sense. They have introduced in their own way , fuzzy equivalence relations and investigated their properties. They obtained results similar to that in the case of ordinary equivalence relations, also the notion of a fuzzy function was introduced , as a special type of fuzzy relation and few of its properties were discussed. The fuzzy function defined by Dib et al [18] became more general than that proposed by Zadeh [65].

In a significant development ,in 2002 Rolly Intan and Masao Mukaidono [27] have published a paper entitled Degree of Similarity in Fuzzy Partition , where they have discussed impreciseness of data in terms of obtaining degree of similarity in which fuzzy set can be used as an alternative . Degree of Similarity between two imprecise data represented by two fuzzy sets is approximately determined by fuzzy conditional probability relation . Moreover they have examined the degree of similarity relation between two fuzzy sets corresponding to fuzzy classes as result of fuzzy partition on a given finite set of data . In addition they have introduced the concept of fuzzy covering as a generalization of fuzzy partition.

The concept of fuzzy similarity relation was used by various researchers for a wide spectrum of investigations . One such paper was found to be on Chromosome Classification and Identification , by M. Elif Karsligil A1, M. Yahya Karsligil A1 [19]. This paper presents a new approach to the classical chromosome classification and identification problem. Its approach maps distinctive features of chromosomes, e.g. length, area, centro mere position and band characteristics , into fuzzy logic membership functions Then fuzzy similarity relations obtained from the membership functions are used to classify and identify the chromosomes. This method has several advantages over classical methods, where usually a pre-built single-criteria of template chromosomes is used to compare the unknown chromosome as to make a decision about its identity. First the formulation of chromosome characteristics using fuzzy logic better compensates for the ambiguities in the shape or band characteristics of chromosome in the metaphase images , second the use of all the characteristics of the chromosomes produce a more fail-safe method . As a preparatory step to the actual identification process they divided chromosomes according to their fuzzy similarity relation based on length and area into groups. To recover from the situations where a chromosome may be misgrouped because of its disconformities to ideal definitions, they refine the grouping of chromosome by applying fuzzy similarity relations which represent the relative Centro-mere positions of chromosomes .

Then the band characteristics of each chromosome in a group has been correlated with the band characteristics of the chromosomes the same group of a preprocessed template to obtain identity of the chromosome. The templates used at this step are updated each time when a chromosome is identified, so the system has an adaptive decision algorithm.

Sourier Sebastian and S.Babu Sunder [56] in their paper entitled ‘Fuzzy Groups and Group Homomorphisms’ have studied the effect of group homomorphisms on the chain of level subgroups of fuzzy groups. They have proved a necessary and sufficient condition under which the chains of level subgroups of homomorphic images (pre-image) of an arbitrary fuzzy group can be obtained from that of a fuzzy group. They claimed that these theorems provide the method to find chains of level subgroups of homomorphic images and pre-images of arbitrary fuzzy groups.

Su-Yun Li, De Gang Chen , Wen Xiang Gu , Hui Wang [42] defined six kinds of fuzzy homomorphisms and studied their in-between relationships. In their paper on fuzzy homomrphisms, they have established their findings in terms of ample examples and propositions.

Prior to these investigations , in 1988 Chowdhury, Chakravorty and Khare [16] gave an alternative definition of fuzzy mapping and fuzzy homomorphism. They have characterized the t-level image and t-level inverse image by corresponding t-level f-fuzzy subgroups. They ended up with a fuzzy analogue of the correspondence theorem between a pair of groups.

The concept of MΓ-group was first put forward by Satyanarayana [55], who , however used the term MΓ-module. G.L Booth and Groenwald [14] obtained some interesting properties of radicals of Γ-near rings. Y.I. Kwon , Y.B. Jun , and J.W.Park [40] have shown in their paper that the usual isomorphism theorems for rings hold good for MΓ-groups . In 1979 , an outstanding work was done by Ravisankar and Shukla [52] in the field of Γ-rings. They introduced the notion of a module over a Γ-ring , which rightly provided a natural setting for Jacobson and Nilradical , an analogue to the ordinary ring theory. Indeed the notion of primitivity was an obvious consequence to their theory.

Very few literatures are available in the corresponding fuzzification of RΓ-modules. A common name in this context is that of Y.B.Jun [25,28,29,30,34,40]. Infact he has worked on this topic both in fuzzy as well as crisp sense , however , in his papers he basically emphasizes on Γ-rings and Γ-near rings in fuzzy setting.

In his book entitled 'Ring Theory' , L.H.Rowen [49] has discussed at length on Tensor Product of Modules. He describes Tensor product as a very important tool for various mathematical developments. According to him , historically it arose from a need to multiply two finite dimensional algebras, however the present day treatment is much more general and elegant and still enables us to recover the earlier construction. Rowen [54], Atiyah [3] have exclusively investigated various properties of Tensor Products of modules.

Normalized fuzzy ideals and fuzzy maximal ideals were defined by S.M.Hong and Y.B.Jun [25] .They have shown that the fuzzy maximal ideals are normalized and take only values {0,1}. Further they obtained a characterization theorem on normalized fuzzy ideals, and investigated few related properties.

In 1979 , D.H.Foster [21] first combined the structure of a fuzzy topological space with that of a fuzzy subgroup (Introduced by Rosenfeld [53]) to formulate the elements of fuzzy topological groups. In 1988 , Bao-Ting Lerner [41] extended this idea on to homomorphic images and inverse images of a fuzzy right topological semigroup.

Back in 1977 , a structure for fuzzy vector spaces and fuzzy topological vector spaces were proposed by Katsaras and Liu [31] , however it took almost ten years , that is in 1987, the very idea of fuzzy topological vector spaces was used by Ferraro and Foster [20] , to study derivation of fuzzy continuous mappings over a fuzzy topological vector space and their various properties. They have also pointed out in their paper that their approach did not depend upon the imposition of a norm on the space, contrary to the earlier findings.

1.2 Outline of the book:

Despite the fact that the literature on Fuzzy Algebra has now become very wide and extensive, being used in all possible ways to almost every branch of algebraic structure , ranging from ordinary groups, rings, vector spaces, modules to the likes of Γ-rings, Γ-semigroups, RΓ- Modules , BCC Algebras, BCK Algebras and many more. Simultaneously the spectrum has also been enlarged , thus giving ample scope to the researchers for finding newer aspects.

This book comprises of seven chapters taken mainly from the results of the studies made by us, presented in our doctoral thesis and published in various journals [8,9,10,11,12,13] . In the study we intend to introduce new concepts ; make them more general ; find their characterizations in terms of the definitions and properties already available in the literature on Fuzzy Algebra . Lot of examples are also being dealt with, in support of our findings .

The main motive of our investigation is to study some virgin concepts, which were yet to be fuzzified and to generalize certain ideas which were already treated with different light and aspects. The key objective of our present study is to open gateways to the avenues of these new and generalized ideas , we have introduced , for their further improvements as well as to carry on further researches in their respective fields.

Chapter 2, comprises of the preliminary ideas , definitions and results , those are pre requisites for the further development of the topics studied in this book .

Chapter 3 , deals with the notion of Γ-semigroup in fuzzy setting . Here we introduce the concept of fuzzy Γ-ideals in a Γ-semigroup by generalizing the definition given by Jun[28]. The concept of (α,β) cut in a Γ-semigroup has been introduced and some interesting results pertaining to (α,β) cut in a Γ-semigroup have been established in section 2.1. A fuzzy Γ-simple semigroup is defined and an equivalence between a simple Γ-semigroup and a fuzzy Γ-simple semigroup is established in section 2.2. We intend to develop in section 2.3, the homologous property of ideals in a Γ-semigroup of fuzzy cosets using the concept of fuzzy homologous subgroups introduced by Zhang et al [66].

In chapter 3, concept of fuzzy similarity relation on $S \times A$ where S is a multiplicatively closed set on a commutative ring with unity A ,and also the notion of α-similarity classes on A are introduced. Section 3.1 presents preliminary ideas pertaining to the concept. A new ring comprising of all such α-similarity classes on A denoted by $S^{-1}A_{\mu}[\alpha]$ under the binary operations of addition and multiplication is obtained in section 3.2 and some of its local properties are discussed in section 3.3. In section 3.4 , we discuss about the shape this new ring takes under primary decomposition .

Chapter 4 deals with various aspects of fuzzy RΓ-submodules and their related properties. In section 4.1, we give the definition of fuzzy RΓ-submodules. Also the notions of faithfull and regular fuzzy submodules are discussed. The notions of fuzzy annihilators, quasi faithful and finitely generated fuzzy RΓ-submodules are introduced and established some of their relevant characteristics in section 4.2 . Concept of R-normal fuzzy subgroups and fuzzy submodules is introduced in section 4.3 and few of their related results are obtained there in.

In chapter 5 , the concept of tensor product of fuzzy submodules over a commutative ring with unity is introduced and its various characteristics are discussed. The tensor product of two R-modules , where R is a Commutative ring with unity has already been an established concept , and also in the discussion of homology and category theory , the concept comes naturally both in crisp as well as fuzzy sense. Here we have identified the situation from a completely different view-point that comes naturally from the algebraic aspect . We try to inspect every such concept developed due to the inception of the tensor product of fuzzy submodules that involve only the algebraic nature . Analogous to the other chapters, here also, section 5.1 is devoted to preliminary results. Section 5.2 presents the definition of tensor product of fuzzy submodules and some of its characteristics. Section 5.3 yields an alternative representation of the tensor product in terms of fuzzy homomorphism.

Chapter 6 , deals with some aspects of fuzzy topological vector space . Here we intend to abridge the concept of a fuzzy topological vector space with that of the fuzzy topological space consisting of all fuzzy subspaces of a given vector space . In section 6.2 , we have obtained that not every element of a fuzzy topological vector space is a fuzzy subspace however under certain