

Instructor's
Resource
Manual

MODERN DIFFERENTIAL EQUATIONS
Theory, Applications, Technology

ABELL & BRASELTON



Instructor's Resource Manual

MODERN DIFFERENTIAL EQUATIONS *Theory, Applications, Technology*

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Saunders College Publishing
HARCOURT BRACE COLLEGE PUBLISHERS

Fort Worth Philadelphia San Diego New York Orlando Austin
San Antonio Toronto Montreal London Sydney Tokyo

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Printed in the United States of America.

Abell & Braselton; Instructor's Resource Manual for Modern Differential Equations, Theory, Applications, Technology, 1E.

ISBN 0-03-016199-1

567 021 987654321

Preface

What is here

The *Instructor's Resource Manual* to accompany *Modern Differential Equations* contains solutions, partial solutions, and hints, including *Mathematica* and *Maple V* code, to all exercises in the text and problems appearing in **In Touch with Technology** and **Differential Equations at Work**. In addition, an **Appendix** reviewing standard techniques of integration is included at the end of the *Manual*.

The Solutions

There is a wide range of detail in the solutions and hints provided in the *Resource Manuals*. Most solutions, especially those that appear in the *Student Resource Manual*, carefully explain the key steps that are encountered in the calculations. Answers to *very* complicated problems were constructed with the help of a computer algebra system.

The Applications

A wide range of application exercises are included throughout the text. Many of the applications are straightforward and involve relatively simple calculations from calculus and differential equations. Others, especially those at the end of the **Chapter Review Exercises** and **Differential Equations at Work** sections are more difficult, may take several hours to complete, and should not be assigned indiscriminately. Instead, they can be assigned as long-term group or individual projects.

The Computer Code

For the most part, the code supplied will generate output from which reasonable conclusions can be made. Due to space limitations, the code presented here is *not* a course in how to use *Mathematica/Maple V* to solve differential equations; the code is not thoroughly explained (see **Other Resources** below). Throughout the *Resource Manuals*, whenever we present computer code, we use the convention that *Mathematica* code appears in the left column in **bold Courier** and *Maple V* code appears in the right column in Monaco. If code for only one computer algebra system is shown, *Mathematica* code appears in **bold Courier** and *Maple V* code appears in Monaco.

Here is some *Mathematica* code.

Here is some *Maple V* code.

The *Mathematica* code was generated using Version 2.2 of *Mathematica* while the *Maple V* code was generated using *Maple V*, Release 3, and an Alpha version of Release 4. As these software packages are updated, we expect that some of the code in these manuals will become obsolete. Consequently, the code that is supplied in the *Student Resource Manual* is currently available on the World Wide Web at

<http://www.cs.gasou.edu/faculty/jimb/diffyq/>

If this sites changes or you are otherwise unable to access the WWW, please contact us (the authors) or the publisher for more up-to-date information. All of the code contained in the *Instructors' Resource Manual* is available to adopters electronically (via e-mail or on disk) directly from us. For information, please contact the publisher. Updates in the code will be made periodically, especially when software is updated. If you notice errors in your calculations (especially if you are using more recent versions of the software than is demonstrated in the manual), please notify us (or, the publisher if you are unable to contact us), especially if the errors you encounter are because of an incompatibility with the version of the software that we have used here to generate these solutions.

We have tried to keep the code presented relatively simple, so that it is easy to understand. Sometimes, we point out features of the computer algebra system being used that are unique, which we later capitalize on. In other cases, we may point out subtle pitfalls or interesting features of the computer algebra system under consideration. Please understand that when we point out an "interesting" feature of a computer algebra system that we can capitalize on here, we are *not* advocating that computer algebra system as the system of choice. Similarly, pointing out a "pitfall"

is *not* stating that the computer algebra system under consideration is inadequate for our purposes, especially since later editions of these software packages may make those observations obsolete. Both computer algebra systems considered in these ancillaries are *very* strong and provide graphic, numeric, and symbolic capabilities that are outstanding for nearly every user. Depending upon your current and future resources, you may prefer one over the other, or another system altogether. In no case, however, should any remark in either the *Instructor's Resource Manual* to accompany *Modern Differential Equations* or the *Students' Resource Manual* to accompany *Modern Differential Equations* be construed for, or against, a particular computer algebra system.

Other Resources

Students and instructors can find substantial guidance in learning how to use computer algebra systems like *Maple V* and *Mathematica* from the wide variety of introductory texts that are available, some of which are listed as follows.

- Abell, Martha L. and Braselton, James P., *Differential Equations with Maple V*, Academic Press Professional, 1994.
- Abell, Martha L. and Braselton, James P., *Differential Equations with Mathematica*, Academic Press Professional, 1992.
- Abell, Martha L. and Braselton, James P., *Maple V By Example*, Academic Press Professional, 1994.
- Abell, Martha L. and Braselton, James P., *Mathematica By Example*, Revised Edition, Academic Press Professional, 1994.
- Blachman, Nancy, *Mathematica: A Practical Approach*, Prentice-Hall, 1992.
- Char, Bruce W., Geddes, Keith O., Gonnet, Gaston H., Leong, Benton L., Monagan, Michael B., and Watt, Stephen M., *First Leaves: A Tutorial Introduction to Maple V*, Springer-Verlag, 1992.
- Wolfram, Stephen, *Mathematica: A System for Doing Mathematics by Computer*, Second Edition, Addison-Wesley, 1991. (A student edition is also available.)

In addition, several journals and newsletters, like those listed below, regularly contain articles that address issues pertaining to the incorporation of technology into college mathematics courses (like college algebra, trigonometry, pre-calculus, calculus, and beyond) as well as the differential equations course.

- CASE (Computer Algebra Systems in Education) Newsletter*, Don Small, Department of Mathematical Sciences, U.S. Military Academy, West Point, New York 10996.
- The C*ODE*E (Consortium for ODE Experiments) Newsletter*, Department of Mathematics, 1250 N. Dartmouth Ave., Harvey Mudd College, Claremont, California 91711.
- The Maple Roots Report* (The Waterloo Maple Software newsletter for *Maple V* users), Waterloo Maple Software, 160 Columbia Street West, Waterloo, Ontario, Canada, E-Mail: info@maplesoft.on.ca.
- MAPLE TECH: The Maple Technical Newsletter*, Birkhäuser, Boston, Service Center Secaucus, 44 Hartz Way, Secaucus, New Jersey, 07094.
- Mathematica in Education*, Paul Wellin, Editor, Department of Mathematics, Sonoma State University, 1801 East Cotati Avenue, Rohnert Park, California, E-Mail: wellin@sonoma.edu.
- The Mathematica Journal*, Miller Freeman, Inc., 600 Harrison Street, San Francisco, California, 94107.
- MathUser* (The Wolfram Research newsletter for *Mathematica* users), Wolfram Research, Inc., 100 Trade Center Drive, Champaign, Illinois, 61820-7237, E-Mail: mathuser@wri.com.

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Contents

1 Introduction to Differential Equations	1
EXERCISES 1.1	1
In Touch with Technology 1.2	1
EXERCISES 1.2	4
In Touch with Technology 1.3	10
EXERCISES 1.3	12
EXERCISES 1.4	14
In Touch with Technology 1.5	25
EXERCISES 1.5	28
CHAPTER 1 REVIEW EXERCISES	29
2 First-Order Ordinary Differential Equations	33
In Touch with Technology 2.1	33
EXERCISES 2.1	37
In Touch with Technology 2.2	51
EXERCISES 2.2	54
In Touch with Technology 2.3	71
EXERCISES 2.3	73
In Touch with Technology 2.4	83
EXERCISES 2.4	86
In Touch with Technology 2.5	105
EXERCISES 2.5	108
In Touch with Technology 2.6	109
EXERCISES 2.6	111
CHAPTER 2 REVIEW EXERCISES	120
Differential Equations at Work: Modeling the Spread of a Disease	132
3 Applications of First-Order Differential Equations	137
In Touch with Technology 3.1	137
EXERCISES 3.1	142
In Touch with Technology 3.2	153
EXERCISES 3.2	155
In Touch with Technology 3.3	159
EXERCISES 3.3	159
In Touch with Technology 3.4	162
EXERCISES 3.4	165
CHAPTER 3 REVIEW EXERCISES	168
Differential Equations at Work: Mathematics of Finance	174
4 Higher-Order Differential Equations	179
In Touch with Technology 4.1	179
EXERCISES 4.1	180
In Touch with Technology 4.2	183
EXERCISES 4.2	185
In Touch with Technology 4.3	191
EXERCISES 4.3	195
In Touch with Technology 4.4	206
EXERCISES 4.4	208
In Touch with Technology 4.5	212
EXERCISES 4.5	215
In Touch with Technology 4.6	229

EXERCISES 4.6	232
In Touch with Technology 4.7	242
EXERCISES 4.7	244
CHAPTER 4 REVIEW EXERCISES	262
Differential Equations at Work: Modeling the Motion of a Skier	273
5 Applications of Higher-Order Equations	281
In Touch with Technology 5.1	281
EXERCISES 5.1	284
In Touch with Technology 5.2	287
EXERCISES 5.2	288
In Touch with Technology 5.3	293
EXERCISES 5.3	297
In Touch with Technology 5.4	305
EXERCISES 5.4	308
In Touch with Technology 5.5	313
EXERCISES 5.5	316
CHAPTER 5 REVIEW EXERCISES	317
Differential Equations at Work: Rack-and-Gear Systems	321
6 Ordinary Differential Equations with Nonconstant Coefficients	323
In Touch with Technology 6.1	323
EXERCISES 6.1	326
In Touch with Technology 6.2	337
EXERCISES 6.2	337
In Touch with Technology 6.3	343
EXERCISES 6.3	345
In Touch with Technology 6.4	367
EXERCISES 6.4	369
In Touch with Technology 6.5	407
EXERCISES 6.5	408
CHAPTER 6 REVIEW EXERCISES	412
Differential Equations at Work: The Schrödinger Equation	424
7 Introduction to the Laplace Transform	427
In Touch with Technology 7.1	427
EXERCISES 7.1	427
In Touch with Technology 7.2	430
EXERCISES 7.2	431
In Touch with Technology 7.3	433
EXERCISES 7.3	433
In Touch with Technology 7.4	436
EXERCISES 7.4	437
In Touch with Technology 7.5	441
EXERCISES 7.5	446
In Touch with Technology 7.6	455
EXERCISES 7.6	456
In Touch with Technology 7.7	459
EXERCISES 7.7	466
CHAPTER 7 REVIEW EXERCISES	481
Differential Equations at Work: The Tautochrone	489
8 Systems of Differential Equations	491
In Touch with Technology 8.1	491
EXERCISES 8.1	498
In Touch with Technology 8.2	512
EXERCISES 8.2	513
In Touch with Technology 8.3	517
EXERCISES 8.3	519
In Touch with Technology 8.4	522
EXERCISES 8.4	528

In Touch with Technology 8.5	538
EXERCISES 8.5	540
In Touch with Technology 8.6	550
EXERCISES 8.6	552
In Touch with Technology 8.7	571
EXERCISES 8.7	574
In Touch with Technology 8.8	581
EXERCISES 8.8	589
CHAPTER 8 REVIEW EXERCISES	595
Differential Equations at Work: Controlling the Spread of a Disease	612
9 Applications of Systems of Ordinary Differential Equations	617
In Touch with Technology 9.1	617
EXERCISES 9.1	622
In Touch with Technology 9.2	627
EXERCISES 9.2	630
In Touch with Technology 9.3	634
EXERCISES 9.3	641
In Touch with Technology 9.4	646
EXERCISES 9.4	648
CHAPTER 9 REVIEW EXERCISES	651
Differential Equations At Work: Free Vibration of a Three-Story Building	655
Appendix: Review of Techniques of Integration	659
Using Tables of Integrals	659
Computer Algebra Systems	660
Integration by Parts	661
Repeated Integration by Parts	662
Integrals Involving Products of Trigonometric Functions	664
Trigonometric Substitution	667
Partial Fractions	671
Distinct (nonrepeated) Linear Factor	671
Repeated Linear Factors	674
Nonrepeated Irreducible Quadratic Factors	674
Repeated Irreducible Quadratic Factors	675

Introduction to Differential Equations

EXERCISES 1.1

- (a) ordinary; (b) second order (c) linear
- (a) ordinary; (b) first-order; (c) nonlinear (in x and y)
- (a) partial; (c) linear
- (a) ordinary; (b) third-order; (c) linear (in y)
- (a) ordinary; (b) first-order; (c) nonlinear
- (a) ordinary; (b) second-order; (c) linear (in y)
- (a) partial; (c) linear
- (a) partial; (b) first-order; (c) nonlinear (in u)
- (a) ordinary; (b) second-order; (c) nonlinear
- (a) ordinary; (b) second-order; (c) nonlinear (in x and t)
- (a) partial; (c) nonlinear
- (a) ordinary; (b) first-order; (c) linear (in y)
- If y is the dependent variable, we write the equation as $\frac{dy}{dx} = 2x - y$ and see that it is (a) ordinary; (b) first-order; and (c) linear. If x is the dependent variable, write it as $\frac{dx}{dy} = \frac{1}{2x - y}$ and see that it is (a) ordinary; (b) first-order; and (c) nonlinear.
- (a) partial; (b) first-order; (c) nonlinear (in u)

- If y is the dependent variable, we write the equation as $\frac{dy}{dx} = \frac{2x - y}{y}$ to see that it is (a) ordinary; (b) first-order; and (c) nonlinear. If x is the dependent variable, we write the equation as $\frac{dx}{dy} = \frac{y}{2x - y}$ to see that it is (a) ordinary; (b) first-order; and (c) nonlinear.
- nonlinear; first-order
- linear; first-order
- linear
- nonlinear
- linear

- (a) $\begin{cases} x' = y \\ y' = y + 6x \end{cases}$; (b) $\begin{cases} x' = y \\ y' = \frac{1}{4}(-4y - 37x) \end{cases}$
- (c) $\begin{cases} x' = y \\ y' = -\frac{g}{L} \sin x \end{cases}$; (d) $\begin{cases} x' = y \\ y' = \mu(1 - x^2)y - x \end{cases}$
- (e) $\begin{cases} x' = y \\ y' = \frac{1}{t}[(t - b)y + ax] \end{cases}$

In Touch with Technology 1.2

- Even though $y = \frac{\sin x}{x}$ is undefined if $x = 0$, both computer algebra systems generate correct graphs.

Mathematica

```
Clear[x, y]
y[x_] = Sin[x]/x;
x y'[x] + y[x] - Cos[x] // Simplify
Plot[y[x], {x, -2Pi, 2Pi}]
```

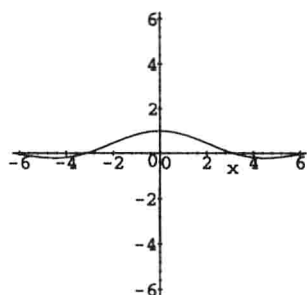
2.

```
y[x_] = Exp[-3x/4] Cos[3x];
16y''[x] + 24y'[x] + 153y[x] // Expand
Plot[y[x], {x, 0, 3Pi/2}]
```

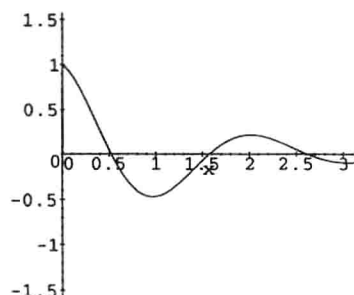
Maple V

```
y:=x->sin(x)/x;
simplify(x*diff(y(x),x)+y(x));
plot(y(x), x=-2*Pi..2*Pi, -2*Pi..2*Pi);
```

```
y:='y':
y:=x->exp(-3*x/4)*cos(3*x);
simplify(16*diff(y(x),x$2)+
24*diff(y(x),x)+153*y(x));
plot(y(x), x=0..Pi, -Pi/2..Pi/2);
```



Graph for Problem 1.



Graph for Problem 2

3.

```

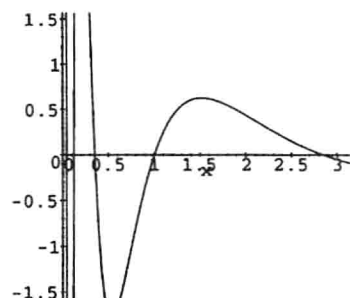
y[x_]=x^(-1) Sin[3 Log[x]]
x^3 y'''[x]+x^2 y''[x]+x y'[x]-
40y[x]//Expand
Plot[y[x], {x, 0, Pi}]

```

```

y:='y':
y:=x->x^(-1)*sin(3*ln(x)):
simplify(x^3*diff(y(x),x$3)+
x^2*diff(y(x),x$2)+x*diff(y(x),x)-
40*y(x));
plot(y(x),x=0..Pi,-Pi/2..Pi/2);

```



4.

```

Clear[x,y,t]
x[t_]=Exp[-t](100Sqrt[3]/3 Sin[Sqrt[3]t]+
20Cos[Sqrt[3] t]);
y[t_]=Exp[-t](-40Sqrt[3]/3 Sin[Sqrt[3]t]+
20Cos[Sqrt[3] t]);
x'[t]-4 y[t]//Simplify
y'[t]-(-x[t]-2y[t])//Simplify

```

```

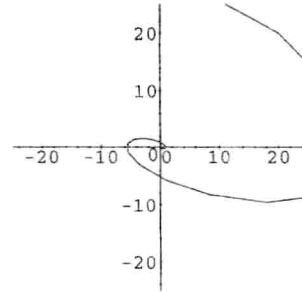
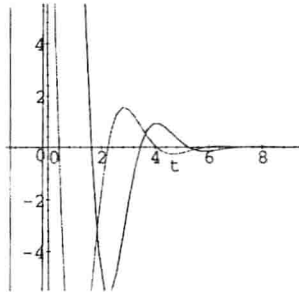
x:='x':y:='y':
x:=t->exp(-t)*
(100*sqrt(3)/3*sin(sqrt(3)*t)+
20*cos(sqrt(3)*t));
y:=t->exp(-t)*
(-40*sqrt(3)/3*sin(sqrt(3)*t)+
20*cos(sqrt(3)*t));
simplify(diff(x(t),t)-4*y(t));
simplify(diff(y(t),t)+x(t)+2*y(t));

```

```

Plot[{x[t],y[t]},{t,0,2Pi},PlotRange->All,
  PlotStyle->{GrayLevel[0],
    Dashing[{0.02}]}]
ParametricPlot[{x[t],y[t]},{t,0,2Pi}]
plot({x(t),y(t)},t=-Pi/2..3*Pi,
  -7*Pi/4..7*Pi/4);
plot([x(t),y(t)],t=-2*Pi..2*Pi,
  view=[-25..25,-25..25]);

```



5.

```

Clear[x,y,eq]
eq=(x^2+y^2)^2==5*x*y;
step1=Dt[eq]
step2=step1 /. {Dt[x]->1,Dt[y]->dydx}
step3=Solve[step2,dydx]

```

```

plot1=ContourPlot[(x^2+y^2)^2-5*x*y,
  {x,-2,2},{y,-2,2},
  Contours->{0},Frame->False,
  ContourShading->False,Axes->Automatic,
  AxesOrigin->{0,0},PlotPoints->60]
plot2=Graphics[{Dashing[{0.02}],
  Line[{{1,-2},{1,2}}],
  Line[{{2,-0.319},{-2,-0.319}}]}];
Show[plot1,plot2]

```

```

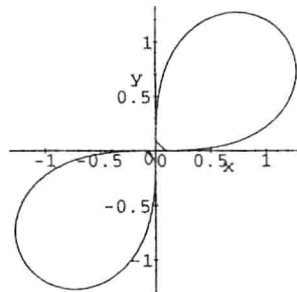
x:='x':y:='y':
step1:=D((x^2+y^2)^2==5*x*y);
step2:=subs({D(x)=1,D(y)=dydx},
  step1);
solve(step2,dydx);

```

```

with(plots):
implicitplot((x^2+y^2)^2==5*x*y,
  x=-2..2,y=-2..2,grid=[45,45]);

```



```

partc=eq /. x->1
Solve[partc]/N
partd=eq /. y->-0.319
Solve[partd]/N

```

```

fsolve((1+y^2)^2==5*y);
fsolve((x^2+(-0.319)^2)^2==
  -5*x*0.319);

```

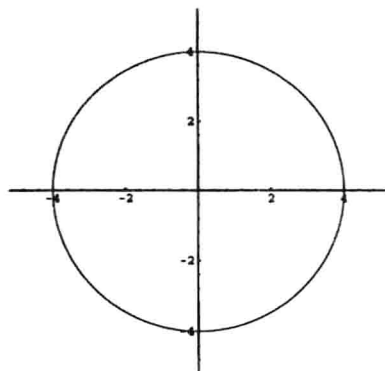
EXERCISES 1.2

- $\frac{dy}{dx} + 2y = (-2e^{-2x}) + 2e^{-2x} = 0$
- $\frac{dy}{dx} = -xe^{-x^2/2} \Rightarrow \frac{dy}{dx} + xy = -xe^{-x^2/2} + xe^{-x^2/2} = 0$
- $\frac{dy}{dx} + y = \left(-e^{-x} + \frac{1}{2}\sin x + \frac{1}{2}\cos x\right) + \left(e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x\right) = \sin x$
- (a) $\frac{dy}{dx} = 4e^{4x}; \frac{d^2y}{dx^2} = 16e^{4x}; \frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 16e^{4x} - 4e^{4x} - 12e^{4x} = 0$
 (b) $\frac{dy}{dx} = -3e^{-3x}; \frac{d^2y}{dx^2} = 9e^{-3x}; \frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 9e^{-3x} - (-3e^{-3x}) - 12e^{-3x} = 0$
- $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} = 81Be^{-9x} + 9(-9Be^{-9x}) = 0$
- $\frac{dx}{dt} = 2Ae^{2t} - 5Be^{-5t}; \frac{d^2x}{dt^2} = 4Ae^{2t} + 25Be^{-5t};$
 $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 10x = (4Ae^{2t} + 25Be^{-5t}) + 3(2Ae^{2t} - 5Be^{-5t}) - 10(Ae^{2t} + Be^{-5t}) = 0$
- $\frac{dx}{dt} = \left(-A + \frac{t}{4} - \frac{1}{2}\right)\sin t + \left(B + \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4}\right)\cos t; \frac{d^2x}{dt^2} = \left(-A + \frac{3t}{4} - 1\right)\cos t + \left(-B - \frac{t^2}{4} + \frac{t}{2}\right)\sin t$
- $\frac{dy}{dx} = 6e^{6x}\cos 2x - 2e^{6x}\sin 2x; \frac{d^2y}{dx^2} = 6e^{6x}\cos 2x - 24e^{6x}\sin 2x$
 $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 40y = 32e^{6x}\cos 2x - 24e^{6x}\sin 2x - 12(6e^{6x}\cos 2x - 2e^{6x}\sin 2x) + 40e^{6x}\cos 2x = 0;$
 $\frac{dy}{dx} = 6e^{6x}\sin 2x + 2e^{6x}\cos 2x; \frac{d^2y}{dx^2} = 32e^{6x}\sin 2x + 24e^{6x}\cos 2x$
 $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 40y = 32e^{6x}\sin 2x + 24e^{6x}\cos 2x - 12(6e^{6x}\sin 2x + 2e^{6x}\cos 2x) + 40e^{6x}\sin 2x = 0$
- $\frac{dy}{dx} = 2Be^{2x} - 2Ce^{-2x}; \frac{d^2y}{dx^2} = 4Be^{2x} + 4Ce^{-2x}; \frac{d^3y}{dx^3} = 8Be^{2x} - 8Ce^{-2x}$
- $\frac{dy}{dx} = B + 2Ce^{2x}; \frac{d^2y}{dx^2} = 4Ce^{2x}; \frac{d^3y}{dx^3} = 8Ce^{2x}; \frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 8Ce^{2x} - 2(4Ce^{2x}) = 0$
- $x^2(30Ax^4 + 42x^5) - 12x(6Ax^5 + 7Bx^6) + 42(Ax^6 + Bx^7) = 0$
- $\frac{dy}{dx} = \frac{-((a-2b)\cos(2\ln x) + (b+2a)\sin(2\ln x))}{x^2}; \frac{d^2y}{dx^2} = \frac{2((-a-3b)\cos(2\ln x) + (3a-3b)\sin(2\ln x))}{x^3};$
 $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = x^2\frac{2((-A-3B)\cos(2\ln x) + (3A-3B)\sin(2\ln x))}{x^3}$
 $+ 3x\frac{-((A-2B)\cos(2\ln x) + (B+2A)\sin(2\ln x))}{x^2} + 5x^{-1}(A\cos(2\ln x) + B\sin(2\ln x)) = 0$

13.

$$2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y};$$

$$0^2 + y^2 = 16 \Rightarrow y = \pm 4; (0, \pm 4)$$



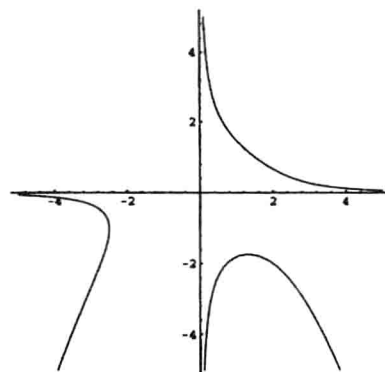
14.

$$3x^2y + x^3 \frac{dy}{dx} + 3y^2 + 6xy \frac{dy}{dx} = 0$$

$$(x^3 + 6xy) \frac{dy}{dx} = -3x^2y - 3y^2$$

$$x(x^2 + 6y)dy + 3y(x^2 + y)dx = 0$$

$$8y + 6y^2 = 8 \Rightarrow 3y^2 + 4y - 4 = 0, y = -2 \text{ or } y = \frac{2}{3}$$

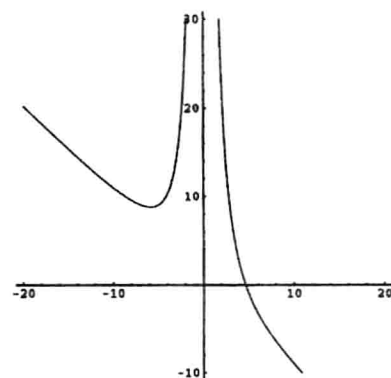


15.

$$3x^2 + 2xy + x^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2};$$

$$1^3 + y = 100 \Rightarrow y = 99; (1, 99)$$

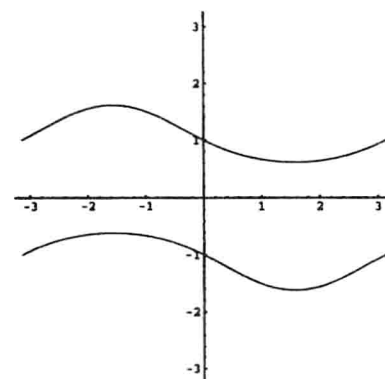


16.

$$2y \frac{dy}{dx} + y \cos x + \frac{dy}{dx} \sin x = 0 \Rightarrow$$

$$(2y + \sin x)dy + y \cos x dx = 0$$

$$y^2 = 1 \Rightarrow y = \pm 1$$

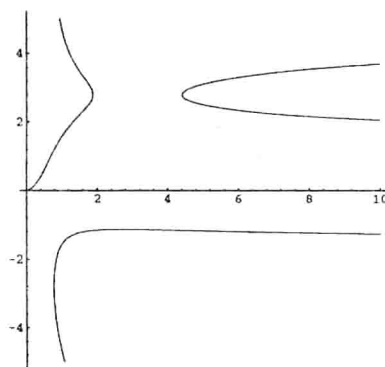


17.

$$\frac{y}{x} + \frac{dy}{dx} \ln x + \cos y - x \sin y \frac{dy}{dx} = 0;$$

$$y \ln 1 + \cos y = 0 \Rightarrow \cos y = 0 \Rightarrow$$

$$y = \frac{(2n+1)\pi}{2}, n = 0, \pm 1, \pm 2, \dots$$



18. Use a u-substitution:

$$\begin{aligned} y &= \int (x^2 - 1)(x^3 - 3x) dx \quad \stackrel{\text{u}}{=} \quad \frac{1}{3} \int u^3 du \\ &\quad u = x^3 - 3x \Rightarrow \frac{1}{3} du = (x^2 - 1) dx \\ &= \frac{1}{12} u^4 + C = \frac{1}{12} (x^3 - 3x)^4 + C = \frac{1}{12} x^{12} - x^{10} + \frac{9}{2} x^8 - 9x^6 + \frac{27}{4} x^4 + C \end{aligned}$$

19. Use a u-substitution:

$$\begin{aligned} y(x) &= \int x \sin(x^2) dx \quad \stackrel{\text{u}}{=} \quad \frac{1}{2} \int \sin u du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos(x^2) + C. \\ &\quad u = x^2 \Rightarrow \frac{1}{2} du = x dx \end{aligned}$$

20. Use a u-substitution:

$$\begin{aligned} y(x) &= \int \frac{x}{\sqrt{x^2 - 16}} dx \quad \stackrel{\text{u}}{=} \quad \frac{1}{2} \int u^{-1/2} du = u^{1/2} + C = \sqrt{x^2 - 16} + C. \\ &\quad u = x^2 - 16 \Rightarrow \frac{1}{2} du = x dx \end{aligned}$$

21. Use a u-substitution:

$$\begin{aligned} y(x) &= \int \frac{1}{x \ln x} dx \quad \stackrel{\text{u}}{=} \quad \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C. \\ &\quad u = \ln x \Rightarrow du = 1/x dx \end{aligned}$$

22. Use integration by parts:

$$\begin{aligned} y(x) &= \int x \ln x dx \quad \stackrel{\text{u}}{=} \quad \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \cdot \frac{1}{x} dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C. \\ &\quad u = \ln x \Rightarrow du = 1/x dx \\ &\quad dv = x dx \Rightarrow v = x^2/2 \end{aligned}$$

23. Use integration by parts:

$$\begin{aligned} y(x) &= \int x e^{-x} dx \quad \stackrel{\text{u}}{=} \quad -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C. \\ &\quad u = x \Rightarrow du = dx \\ &\quad dv = e^{-x} dx \Rightarrow v = -e^{-x} \end{aligned}$$

24. Use partial fractions:

$$y(x) = \int \frac{-2(x+5)}{(x+2)(x-4)} dx = \int \left(\frac{1}{x+2} - \frac{3}{x-4} \right) dx = \ln|x+2| - 3 \ln|x-4| + C.$$

25. Use partial fractions. First,

$$\frac{x - x^2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2 + (B+C)x + A+C}{(x+1)(x^2+1)}.$$

Equating coefficients of like terms yields the system of equations

$$\begin{cases} A+B=-1 \\ B+C=1 \\ A+C=0 \end{cases} \quad \stackrel{\text{Subtract (2) from (1)}}{\Rightarrow} \quad \begin{cases} A-C=-2 \\ A+C=0 \end{cases} \quad \stackrel{\text{Add (1) and (2)}}{\Rightarrow} \quad 2A=-2 \Rightarrow A=-1, C=1, B=0.$$

Thus,

$$y(x) = \int \frac{x - x^2}{(x+1)(x^2+1)} dx = \int \left[-\frac{1}{x+1} + \frac{1}{x^2+1} \right] dx = -\ln|x+1| + \tan^{-1} x + C.$$

26. Use a trigonometric substitution with $x = 4 \sec \theta$:

$$\begin{aligned} y(x) &= \int \frac{\sqrt{x^2 - 16}}{x} dx \quad \begin{array}{l} \stackrel{x=4 \sec \theta}{=} \\ dx=4 \sec \theta \tan \theta d\theta \end{array} \quad \int \frac{\sqrt{(4 \sec \theta)^2 - 16}}{4 \sec \theta} 4 \sec \theta \tan \theta d\theta \\ &= \int \frac{4 \tan \theta}{4 \sec \theta} 4 \sec \theta \tan \theta d\theta = 4 \int \tan^2 \theta d\theta = 4 \int (\sec^2 \theta - 1) d\theta \\ &= 4 \tan \theta - 4\theta + C \quad \begin{array}{l} \stackrel{x=4 \sec \theta \Rightarrow \theta = \sec^{-1} \frac{x}{4}}{=} \\ 4 \tan \theta = \sqrt{16 \sec^2 \theta - 16} = \sqrt{x^2 - 16} \end{array} \quad \sqrt{x^2 - 16} - 4 \sec^{-1} \frac{x}{4} + C. \end{aligned}$$

27. Use a trigonometric substitution with $x = 2 \sin \theta$ and the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ (twice):

$$\begin{aligned} y(x) &= \int (4 - x^2)^{3/2} dx \quad \begin{array}{l} \stackrel{x=2 \sin \theta}{=} \\ dx=2 \cos \theta d\theta \end{array} \quad \int (4 - (2 \sin \theta)^2)^{3/2} \cdot 2 \cos \theta d\theta \\ &= \int (4 \cos^2 \theta)^{3/2} \cdot 2 \cos \theta d\theta = 16 \int \cos^4 \theta d\theta \\ &= 16 \int \left[\frac{1}{2}(1 + \cos 2\theta) \right]^2 d\theta = 4 \int (1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta \\ &= 4 \int \left[1 + 2 \cos 2\theta + \frac{1}{2}(1 + \cos 4\theta) \right] d\theta = 4 \int \left(\frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta \\ &= 4 \left(\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right) + C. \end{aligned}$$

Now, $x = 2 \sin \theta$ so $\theta = \sin^{-1} \left(\frac{x}{2} \right)$. Also, $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{x^2}{4}} = \frac{\sqrt{4 - x^2}}{2}$ so

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4 - x^2}}{2} = \frac{1}{2} x \sqrt{4 - x^2}.$$

Similarly,

$$\cos 2\theta = \frac{\cos^2 \theta - \sin^2 \theta}{2 \cos^2 \theta - 1} = \frac{2 - x^2}{2}$$

so

$$\sin 4\theta = 2 \sin 2\theta \cos 2\theta = 2 \cdot \frac{1}{2} x \sqrt{4 - x^2} \cdot \frac{2 - x^2}{2} = \frac{1}{2} x (2 - x^2) \sqrt{4 - x^2}.$$

Thus,

$$\begin{aligned} y(x) &= 4 \left(\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right) + C \\ &= 4 \left(\frac{3}{2} \sin^{-1} \left(\frac{x}{2} \right) + \frac{1}{2} x \sqrt{4 - x^2} + \frac{1}{8} \cdot \frac{1}{2} x (2 - x^2) \sqrt{4 - x^2} \right) + C \\ &= 6 \sin^{-1} \left(\frac{x}{2} \right) + 2x \sqrt{4 - x^2} + \frac{1}{4} x (2 - x^2) \sqrt{4 - x^2} + C. \end{aligned}$$

$$28. \frac{dS}{dt} = -\frac{3(15,000,000)}{(t+100)^4}, \frac{dS}{dt} + \frac{3}{t+100}S = -\frac{3(15,000,000)}{(t+100)^4} + \frac{3(15,000,000)}{(t+100)^4} = 0, S(0) = 15, \lim_{t \rightarrow \infty} S(t) = 0$$

$$29. x(0) = 3; \frac{dx}{dt} = -12 \sin 4t + 9 \cos 4t, \frac{dx}{dt}(0) = 9$$

$$30. u_{xx} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, u_{xx} + u_{yy} = 0$$

$$31. u_t = 16ke^{-16t} \cos 4x, u_{xx} = 16e^{-16t} \cos 4x; u(\pi, 0) = 2; \lim_{t \rightarrow \infty} u(x, t) = 3$$

$$32. u_{tt} = -\cos t \sin \pi x, u_{xx} = -\pi^2 \cos t \sin \pi x, u(0, t) = u(1, t) = 0$$

$$33. \text{ If } y = x^m, y' = mx^{m-1} \text{ and } y'' = m(m-1)x^{m-2}. \text{ Substitution into the equation yields}$$

$$x^2 y'' - 2xy' + 2y = 0$$

$$x^2 \cdot m(m-1)x^{m-2} - 2x \cdot mx^{m-1} + 2x^m = 0$$

$$m(m-1)x^m - 2mx^m + 2x^m = 0$$

$$x^m [m(m-1) - 2m + 2] = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

so $m = 1, m = 2$.

$$34. \text{ If } y = e^{mx}, y' = me^{mx} \text{ and } y'' = m^2 e^{mx}. \text{ Substitution into the equation yields}$$

$$y'' - 3y' - 18y = 0$$

$$m^2 e^{mx} - 3me^{mx} - 18e^{mx} = 0$$

$$e^{mx} (m^2 - 3m - 18) = 0$$

$$m^2 - 3m - 18 = 0$$

$$(m+3)(m-6) = 0$$

so $m = -3, m = 6$.

35.

$$e^{2x} \frac{dy}{dx} + 2e^{2x} y = e^x$$

$$\frac{d}{dx} (e^{2x} y) = e^x$$

$$e^{2x} y = \int e^x dx = e^x + C$$

$$y = \frac{e^x + C}{e^{2x}} = e^{-x} + Ce^{-2x}$$

36.

$$\begin{aligned}
 e^x \frac{dy}{dx} + e^x y &= x e^x \\
 \frac{d}{dx}(e^x y) &= x e^x \\
 e^x y &= \int x e^x dx \stackrel{\substack{\text{Integration by parts with} \\ u=x \Rightarrow du=dx \\ \text{and} \\ dv=e^x dx \Rightarrow v=e^x}}{=} x e^x - \int e^x dx = x e^x - e^x + C \\
 y &= \frac{x e^x - e^x + C}{e^x} = -1 + x + C e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 37. \text{ Because } \frac{d\psi(x)}{dx} &= \frac{An\pi}{L} \cos\left(\frac{n\pi x}{L}\right) \text{ and } \frac{d^2\psi(x)}{dx^2} = -\frac{An^2\pi^2}{L^2} \sin\left(\frac{n\pi x}{L}\right), \\
 -\frac{h^2}{2m} \frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) &= \frac{h^2}{2m} \frac{An^2\pi^2}{L^2} \sin\left(\frac{n\pi x}{L}\right) = EA \sin\left(\frac{n\pi x}{L}\right).
 \end{aligned}$$

$$\text{Therefore, } E = \frac{h^2 n^2 \pi^2}{2mL^2}.$$

38.

$$\begin{aligned}
 -\frac{4}{x^3} + \frac{1}{x^2} + \frac{2}{y^3} \frac{dy}{dx} - \frac{1}{y^2} \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= \frac{(x-4)y^3}{x^3(y-2)}; \text{ yes; yes}
 \end{aligned}$$

39. We use implicit differentiation:

$$\begin{aligned}
 1 + \frac{2x}{y} - \frac{x^2}{y^2} \frac{dy}{dx} &= 0 \\
 -\frac{x^2}{y^2} \frac{dy}{dx} &= -1 - \frac{2x}{y} \\
 \frac{dy}{dx} &= \frac{-1 - 2x/y}{-x^2/y^2} \\
 \frac{dy}{dx} &= \frac{y^2 + 2xy}{x^2}.
 \end{aligned}$$

Solving $x + \frac{x^2}{y} = C$ for y yields $y = \frac{x^2}{C-x}$. Note that $y=0$, which is a solution of $\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$, cannot be

obtained from $y = \frac{x^2}{C-x}$ so is a singular solution.

$$\begin{aligned}
 40. \quad x'(t) &= 4e^t \sin t - 2e^t \cos t, \quad y'(t) = -e^t \cos t - 3e^t \sin t, \\
 -2y(t) &= 4e^t \sin t - 2e^t \cos t, \quad x(t) + 2y(t) = -e^t \cos t - 3e^t \sin t
 \end{aligned}$$

41.

$$\begin{aligned}
 &a[C_1 y_1''(x) + C_2 y_2''(x)] + b[C_1 y_1'(x) + C_2 y_2'(x)] + c[C_1 y_1(x) + C_2 y_2(x)] \\
 &= C_1[ay_1''(x) + by_1'(x) + cy_1(x)] + C_2[ay_2''(x) + by_2'(x) + cy_2(x)] \\
 &= C_1(0) + C_2(0) = 0
 \end{aligned}$$