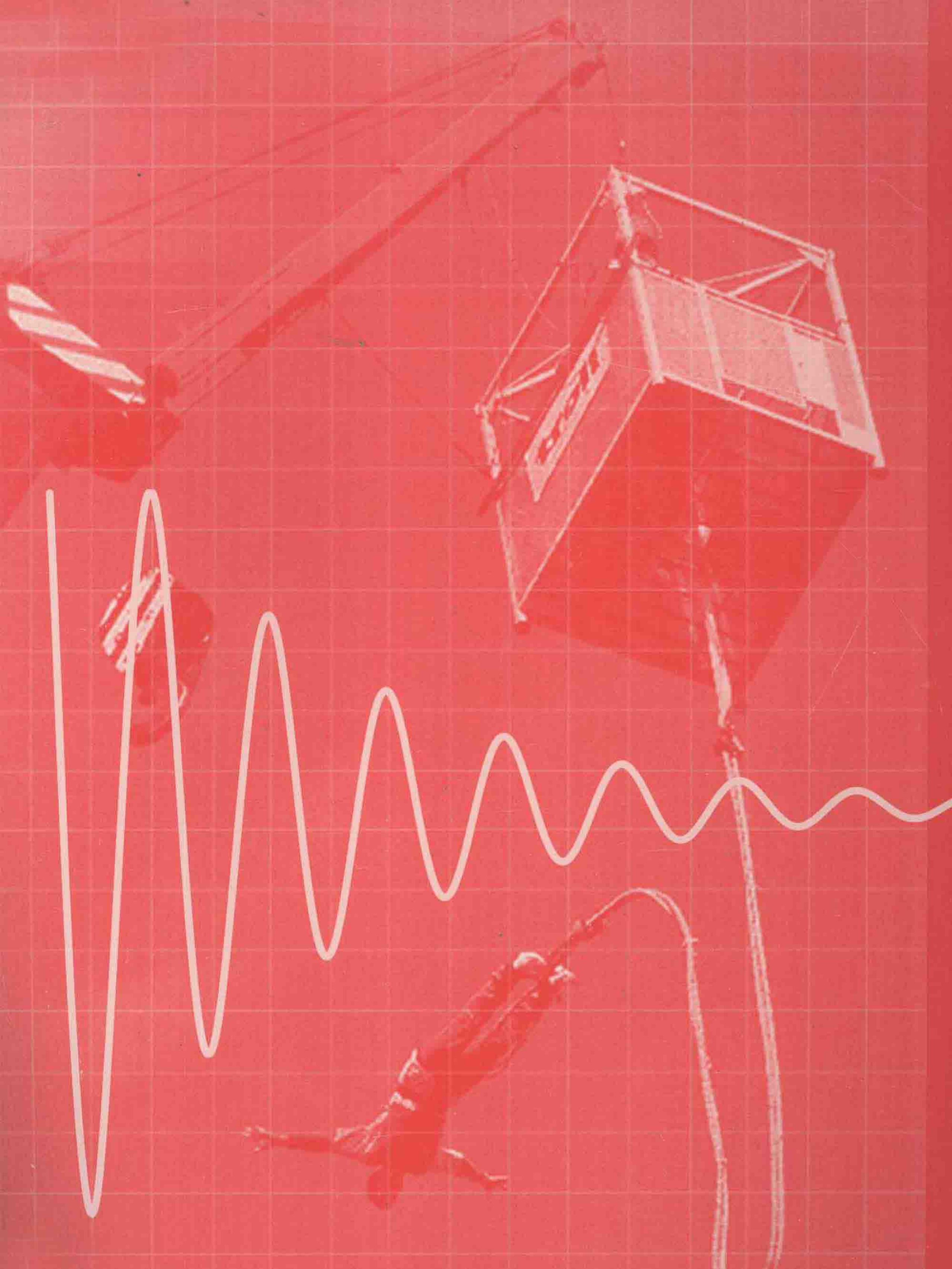
Instructor's Resource Manual

MODERN DIFFERENTIAL EQUATIONS

Theory, Applications, Technology



ABELL & BRASELTON

Instructor's Resource Manual

MODERN DIFFERENTIAL EQUATIONS

Theory, Applications, Technology

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Preface

What is here

The *Instructor's Resource Manual* to accompany *Modern Differential Equations* contains solutions, partial solutions, and hints, including *Mathematica* and *Maple V* code, to all exercises in the text and problems appearing in **In Touch with Technology** and **Differential Equations at Work**. In addition, an **Appendix** reviewing standard techniques of integration is included at the end of the *Manual*.

The Solutions

There is a wide range of detail in the solutions and hints provided in the *Resource Manuals*. Most solutions, especially those that appear in the *Student Resource Manual*, carefully explain the key steps that are encountered in the calculations. Answers to *very* complicated problems were constructed with the help of a computer algebra system.

The Applications

A wide range of application exercises are included throughout the text. Many of the applications are straightforward and involve relatively simple calculations from calculus and differential equations. Others, especially those at the end of the **Chapter Review Exercises** and **Differential Equations at Work** sections are more difficult, may take several hours to complete, and should not be assigned indiscriminately. Instead, they can be assigned as long-term group or individual projects.

The Computer Code

For the most part, the code supplied will generate output from which reasonable conclusions can be made. Due to space limitations, the code presented here is ***not*** a course in how to use *Mathematica*/*Maple V* to solve differential equations; the code is not thoroughly explained (see **Other Resources** below). Throughout the *Resource Manuals*, whenever we present computer code, we use the convention that *Mathematica* code appears in the left column in **`bold Courier`** and *Maple V* code appears in the right column in `Monaco`. If code for only one computer algebra system is shown, *Mathematica* code appears in **`bold Courier`** and *Maple V* code appears in `Monaco`.

```
Here is some Mathematica code.          Here is some Maple V code.
```

The *Mathematica* code was generated using Version 2.2 of *Mathematica* while the *Maple V* code was generated using *Maple V*, Release 3, and an Alpha version of Release 4. As these software packages are updated, we expect that some of the code in these manuals will become obsolete. Consequently, the code that is supplied in the *Student Resource Manual* is currently available on the World Wide Web at

http://www.cs.gasou.edu/faculty/jimb/diffyq/

If this sites changes or you are otherwise unable to access the WWW, please contact us (the authors) or the publisher for more up-to-date information. All of the code contained in the *Instructors' Resource Manual* is available to adopters electronically (via e-mail or on disk) directly from us. For information, please contact the publisher. Updates in the code will be made periodically, especially when software is updated. If you notice errors in your calculations (especially if you are using more recent versions of the software than is demonstrated in the manual), please notify us (or, the publisher if you are unable to contact us), especially if the errors you encounter are because of an incompatibility with the version of the software that we have used here to generate these solutions.

We have tried to keep the code presented relatively simple, so that it is easy to understand. Sometimes, we point out features of the computer algebra system being used that are unique, which we later capitalize on. In other cases, we may point out subtle pitfalls or interesting features of the computer algebra system under consideration. Please understand that when we point out an "interesting" feature of a computer algebra system that we can capitalize on here, we are ***not*** advocating that computer algebra system as the system of choice. Similarly, pointing out a "pitfall" is ***not*** stating that the computer algebra system under consideration is inadequate for our purposes, especially since later editions of these software packages may make those observations obsolete. Both computer algebra systems considered in these ancillaries are ***very*** strong and provide graphic, numeric, and symbolic capabilities that are outstanding for nearly every user. Depending upon your current and future resources, you may prefer one over the other, or another system altogether. In no case, however, should any remark in either the *Instructor's Resource Manual* to accompany *Modern Differential Equations* or the *Students' Resource Manual* to accompany *Modern Differential Equations* be construed for, or against, a particular computer algebra system.

Other Resources

Students and instructors can find substantial guidance in learning how to use computer algebra systems like *Maple V* and *Mathematica* from the wide variety of introductory texts that are available, some of which are listed as follows.

Abell , Martha L. and Braselton, James P., *Differential Equations with Maple V*, Academic Press Professional, 1994.

Abell , Martha L. and Braselton, James P., *Differential Equations with Mathematica*, Academic Press Professional, 1992.

Abell , Martha L. and Braselton, James P., *Maple V By Example*, Academic Press Professional, 1994.

Abell , Martha L. and Braselton, James P., *Mathematica By Example*, Revised Edition, Academic Press Professional, 1994.

Blachman, Nancy, *Mathematica: A Practical Approach*, Prentice-Hall, 1992.

Char, Bruce W., Geddes, Keith O., Gonnet, Gaston H., Leong, Benton L., Monagan, Michael B., and Watt, Stephen M., *First Leaves: A Tutorial Introduction to Maple V*, Springer-Verlag, 1992.

Wolfram, Stephen, *Mathematica: A System for Doing Mathematics by Computer*, Second Edition, Addison-Wesley, 1991. (A student edition is also avalailable.)

In addition, several journals and newsletters, like those listed below, regularly contain articles that address issues pertaining to the incorporation of technology into college mathematics courses (like college algebra, trigonometry, pre-calculus, calculus, and beyond) as well as the differential equations course.

CASE (Computer Algebra Systems in Education) Newsletter, Don Small, Department of Mathematical Sciences, U.S. Military Academy, West Point, New York 10996.

*The C*ODE*E (Consortium for ODE Experiments) Newsletter*, Department of Mathematics, 1250 N. Dartmouth Ave., Harvey Mudd College, Claremont, California 91711.

The Maple Roots Report (The Waterloo Maple Software newsletter for *Maple V* users), Waterloo Maple Software, 160 Columbia Street West, Waterloo, Ontario, Canada, E-Mail: info@maplesoft.on.ca.

MAPLE TECH: The Maple Technical Newsletter, Birkhäuser, Boston, Service Center Secaucus, 44 Hartz Way, Secaucus, New Jersey, 07094.

Mathematica in Education, Paul Wellin, Editor, Department of Mathematics, Sonoma State University, 1801 East Cotati Avenue, Rohnert Park, California, E-Mail: wellin@sonoma.edu.

The Mathematica Journal, Miller Freeman, Inc., 600 Harrison Street, San Francisco, California, 94107.

MathUser (The Wolfram Research newsletter for *Mathematica* users), Wolfram Research, Inc., 100 Trade Center Drive, Champaign, Illimois, 61820-7237, E-Mail: mathuser@wri.com.

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Contents

1

Introduction to Differential Equations

EXERCISES 1.1

1. (a) ordinary; (b) second order (c) linear
2. (a) ordinary; (b) first-order; (c) nonlinear (in x and y)
3. (a) partial; (c) linear
4. (a) ordinary; (b) third-order; (c) linear (in y)
5. (a) ordinary; (b) first-order; (c) nonlinear
6. (a) ordinary; (b) second-order; (c) linear (in y)
7. (a) partial; (c) linear
8. (a) partial; (b) first-order; (c) nonlinear (in u)
9. (a) ordinary; (b) second-order; (c) nonlinear
10. (a) ordinary; (b) second-order; (c) nonlinear (in x and t)
11. (a) partial; (c) nonlinear
12. (a) ordinary; (b) first-order; (c) linear (in y)
13. If y is the dependent variable, we write the equation as $\frac{dy}{dx} = 2x - y$ and see that it is (a) ordinary; (b) first-order; and (c) linear. If x is the dependent variable, write it as $\frac{dx}{dy} = \frac{1}{2x - y}$ and see that it is (a) ordinary; (b) first-order; and (c) nonlinear.
14. (a) partial; (b) first-order; (c) nonlinear (in u)
15. If y is the dependent variable, we write the equation as $\frac{dy}{dx} = \frac{2x - y}{y}$ to see that it is (a) ordinary; (b) first-order; and (c) nonlinear. If x is the dependent variable, we write the equation as $\frac{dx}{dy} = \frac{y}{2x - y}$ to see that it is (a) ordinary; (b) first-order; and (c) nonlinear.
16. nonlinear; first-order
17. linear; lirst-lrder
18. linear
19. nonlinear
20. linear
21. (a) $\begin{cases} x' = y \\ y' = y + 6x \end{cases}$; (b) $\begin{cases} x' = y \\ y' = \frac{1}{4}(-4y - 37x) \end{cases}$;

(c) $\begin{cases} x' = y \\ y' = -\frac{g}{L}\sin x \end{cases}$; (d) $\begin{cases} x' = y \\ y' = \mu\left(1 - x^2\right)y - x \end{cases}$;

(e) $\begin{cases} x' = y \\ y' = \frac{1}{t}\left[(t - b)y + ax\right] \end{cases}$

In Touch with Technology 1.2

1. Even though $y = \frac{\sin x}{x}$ is undefined if $x = 0$, both computer algebra systems generate correct graphs.

Mathematica

```
Clear[x,y]
y[x_]=Sin[x]/x;
x y'[x]+y[x]-Cos[x]//Simplify
Plot[y[x],{x,-2Pi,2Pi}]
```

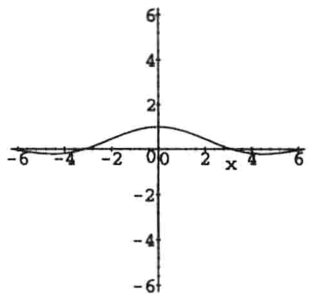
Maple V

```
y:=x->sin(x)/x:
simplify(x*diff(y(x),x)+y(x));
plot(y(x),x=-2*Pi..2*Pi,-2*Pi..2*Pi);
```

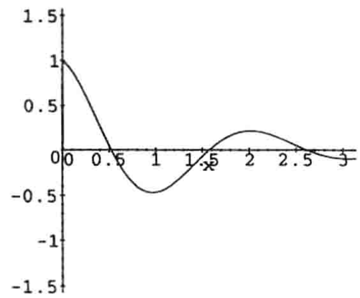
2.

```
y[x_]=Exp[-3x/4]Cos[3x];
16y''[x]+24y'[x]+153y[x]//Expand
Plot[y[x],{x,0,3Pi/2}]
```

```
y:='y':
y:=x->exp(-3*x/4)*cos(3*x):
simplify(16*diff(y(x),x$2)+
   24*diff(y(x),x)+153*y(x));
plot(y(x),x=0..Pi,-Pi/2..Pi/2);
```



Graph for Problem 1.

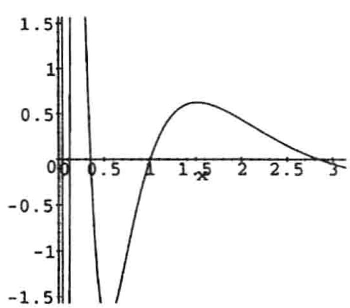


Graph for Problem 2

3.

```
y[x_]=x^(-1) Sin[3 Log[x]]
x^3 y'''[x]+x^2 y''[x]+x y'[x]-
    40y[x]//Expand
Plot[y[x],{x,0,Pi}]
```

```
y:='y':
y:=x->x^(-1)*sin(3*ln(x)):
simplify(x^3*diff(y(x),x$3)+
    x^2*diff(y(x),x$2)+x*diff(y(x),x)-
    40*y(x));
plot(y(x),x=0..Pi,-Pi/2..Pi/2);
```



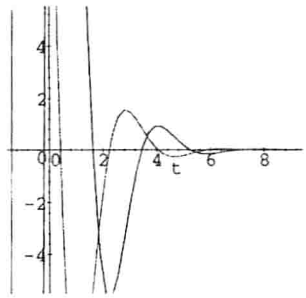
4.

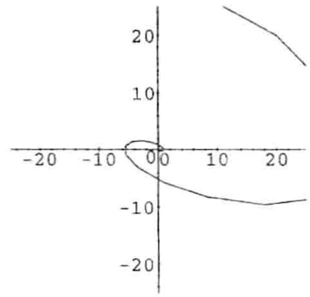
```
Clear[x,y,t]
x[t_]=Exp[-t](100Sqrt[3]/3 Sin[Sqrt[3]t]+
    20Cos[Sqrt[3] t]);
y[t_]=Exp[-t](-40Sqrt[3]/3 Sin[Sqrt[3]t]+
    20Cos[Sqrt[3] t]);
x'[t]-4 y[t]//Simplify
y'[t]-(-x[t]-2y[t])//Simplify
```

```
x:='x':y:='y':
x:=t->exp(-t)*
    (100*sqrt(3)/3*sin(sqrt(3)*t)+
        20*cos(sqrt(3)*t)):
y:=t->exp(-t)*
    (-40*sqrt(3)/3*sin(sqrt(3)*t)+
        20*cos(sqrt(3)*t)):
simplify(diff(x(t),t)-4*y(t));
simplify(diff(y(t),t)+x(t)+2*y(t));
```

```
Plot[{x[t],y[t]},{t,0,2Pi},PlotRange->All,
    PlotStyle->{GrayLevel[0],
        Dashing[{0.02}]}]
ParametricPlot[{x[t],y[t]},{t,0,2Pi}]
```

```
plot({x(t),y(t)},t=-Pi/2..3*Pi,
    -7*Pi/4..7*Pi/4);
plot([x(t),y(t),t=-2*Pi..2*Pi],
    view=[-25..25,-25..25]);
```





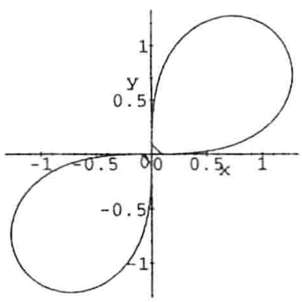
5.

```
Clear[x,y,eq]
eq=(x^2+y^2)^2==5x y;
step1=Dt[eq]
step2=step1 /. {Dt[x]->1,Dt[y]->dydx}
step3=Solve[step2,dydx]
```

```
x:='x':y:='y':
step1:=D((x^2+y^2)^2=5*x*y);
step2:=subs({D(x)=1,D(y)=dydx},
    step1);
solve(step2,dydx);
```

```
plot1=ContourPlot[(x^2+y^2)^2-5x y,
    {x,-2,2},{y,-2,2},
    Contours->{0},Frame->False,
    ContourShading->False,Axes->Automatic,
    AxesOrigin->{0,0},PlotPoints->60]
plot2=Graphics[{Dashing[{0.02}],
    Line[{{1,-2},{1,2}}],
    Line[{{2,-0.319},{-2,-0.319}}]}];
Show[plot1,plot2]
```

```
with(plots):
implicitplot((x^2+y^2)^2=5*x*y,
    x=-2..2,y=-2..2,grid=[45,45]);
```




```
partc=eq /. x->1
Solve[partc]//N
partd=eq /. y->-0.319
Solve[partd]//N
```

```
fsolve((1+y^2)^2=5*y);
fsolve((x^2+(-0.319)^2)^2=
    -5*x*0.319);
```

EXERCISES 1.2

1. $\frac{dy}{dx}+2y=\left(-2e^{-2x}\right)+2e^{-2x}=0$

2. $\frac{dy}{dx}=-xe^{-x^2/2} \Rightarrow \frac{dy}{dx}+xy=-xe^{-x^2/2}+xe^{-x^2/2}=0$

3. $\frac{dy}{dx}+y=\left(-e^{-x}+\frac{1}{2}\sin x+\frac{1}{2}\cos x\right)+\left(e^{-x}-\frac{1}{2}\cos x+\frac{1}{2}\sin x\right)=\sin x$

4. (a) $\frac{dy}{dx}=4e^{4x};\frac{d^2y}{dx^2}=16e^{4x};\frac{d^2y}{dx^2}-\frac{dy}{dx}-12y=16e^{4x}-4e^{4x}-12e^{4x}=0$

(b) $\frac{dy}{dx}=-3e^{-3x};\frac{d^2y}{dx^2}=9e^{-3x};\frac{d^2y}{dx^2}-\frac{dy}{dx}-12y=9e^{-3x}-\left(-3e^{-3x}\right)-12e^{-3x}=0$

5. $\frac{d^2y}{dx^2}+9\frac{dy}{dx}=81Be^{-9x}+9\left(-9Be^{-9x}\right)=0$

6. $\frac{dx}{dt}=2Ae^{2t}-5Be^{-5t},\ \frac{d^2x}{dt^2}=4Ae^{2t}+25Be^{-5t};$

$\frac{d^2x}{dt^2}+3\frac{dx}{dt}-10x=\left(4Ae^{2t}+25Be^{-5t}\right)+3\left(2Ae^{2t}-5Be^{-5t}\right)-10\left(Ae^{2t}+Be^{-5t}\right)=0$

7. $\frac{dx}{dt}=\left(-A+\frac{t}{4}-\frac{1}{2}\right)\sin t+\left(B+\frac{t^2}{4}-\frac{t}{2}+\frac{1}{4}\right)\cos t;\frac{d^2x}{dt^2}=\left(-A+\frac{3t}{4}-1\right)\cos t+\left(-B-\frac{t^2}{4}+\frac{t}{2}\right)\sin t$

8. $\frac{dy}{dx}=6e^{6x}\cos 2x-2e^{6x}\sin 2x;\frac{d^2y}{dx^2}=6e^{6x}\cos 2x-24e^{6x}\sin 2x$

$\frac{d^2y}{dx^2}-12\frac{dy}{dx}+40y=32e^{6x}\cos 2x-24e^{6x}\sin 2x-12\left(6e^{6x}\cos 2x-2e^{6x}\sin 2x\right)+40e^{6x}\cos 2x=0;$

$\frac{dy}{dx}=6e^{6x}\sin 2x+2e^{6x}\cos 2x;\frac{d^2y}{dx^2}=32e^{6x}\sin 2x+24e^{6x}\cos 2x$

$\frac{d^2y}{dx^2}-12\frac{dy}{dx}+40y=32e^{6x}\sin 2x+24e^{6x}\cos 2x-12\left(6e^{6x}\sin 2x+2e^{6x}\cos 2x\right)+40e^{6x}\sin 2x=0$

9. $\frac{dy}{dx}=2Be^{2x}-2Ce^{-2x};\frac{d^2y}{dx^2}=4Be^{2x}+4Ce^{-2x};\frac{d^3y}{dx^3}=8Be^{2x}-8Ce^{-2x}$

10. $\frac{dy}{dx}=B+2Ce^{2x},\frac{d^2y}{dx^2}=4Ce^{2x},\frac{d^3y}{dx^3}=8Ce^{2x};\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}=8Ce^{2x}-2\left(4Ce^{2x}\right)=0$

11. $x^2\left(30Ax^4+42x^5\right)-12x\left(6Ax^5+7Bx^6\right)+42\left(Ax^6+Bx^7\right)=0$

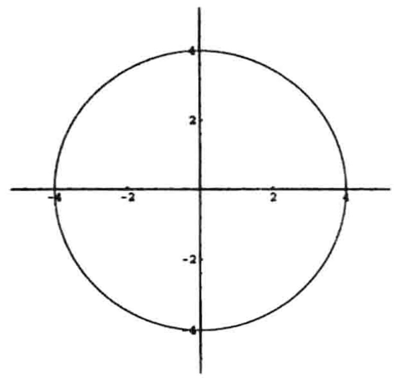
12. $\frac{dy}{dx}=\frac{-((a-2b)\cos(2\ln x)+(b+2a)\sin(2\ln x))}{x^2},\ \frac{d^2y}{dx^2}=\frac{2((-a-3b)\cos(2\ln x)+(3a-3b)\sin(2\ln x))}{x^3},$

$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+5y=x^2\frac{2((-A-3B)\cos(2\ln x)+(3A-3B)\sin(2\ln x))}{x^3}$

$+3x\frac{-((A-2B)\cos(2\ln x)+(B+2A)\sin(2\ln x))}{x^2}+5x^{-1}(A\cos(2\ln x)+B\sin(2\ln x))=0$

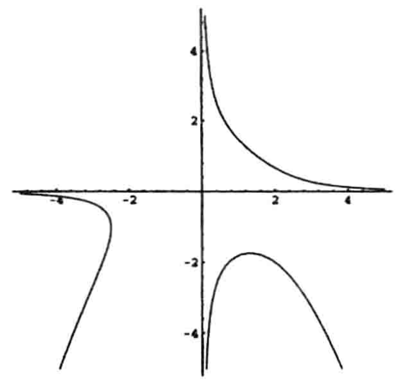
13.

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y};$$
$$0^2 + y^2 = 16 \Rightarrow y = \pm 4; (0, \pm 4)$$



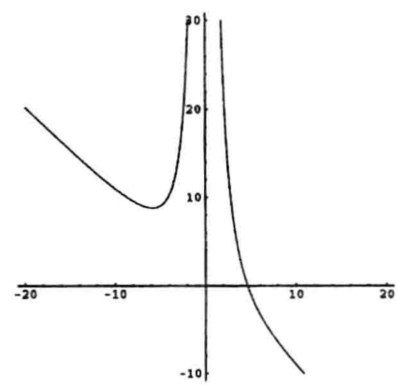
14.

$$3x^2y + x^3\frac{dy}{dx} + 3y^2 + 6xy\frac{dy}{dx} = 0$$
$$\left(x^3 + 6xy\right)\frac{dy}{dx} = -3x^2y - 3y^2$$
$$x\left(x^2 + 6y\right)dy + 3y\left(x^2 + y\right)dx = 0$$
$$8y + 6y^2 = 8 \Rightarrow 3y^2 + 4y - 4 = 0, \ y = -2 \text{ or } y = \frac{2}{3}$$



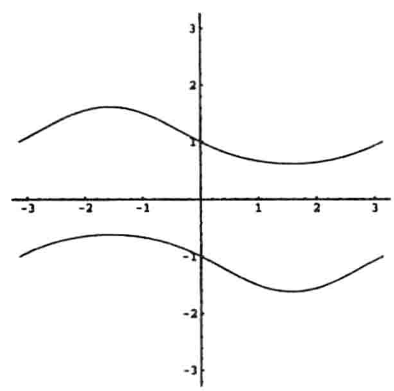
15.

$$3x^2 + 2xy + x^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2};$$
$$1^3 + y = 100 \Rightarrow y = 99; (1, 99)$$



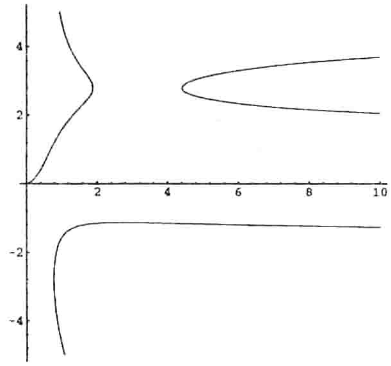
16.

$$2y\frac{dy}{dx} + y\cos x + \frac{dy}{dx}\sin x = 0 \Rightarrow$$
$$(2y + \sin x)dy + y\cos x dx = 0$$
$$y^2 = 1 \Rightarrow y = \pm 1$$



17.

$$\frac{y}{x}+\frac{dy}{dx}\ln x+\cos y-x\sin y\frac{dy}{dx}=0;$$
$$y\ln 1+\cos y=0\Rightarrow \cos y=0\Rightarrow$$
$$y=\frac{(2n+1)\pi}{2},\ n=0,\pm 1,\pm 2,\ldots$$



18. Use a u-substitution:

$$y=\int\left(x^2-1\right)\left(x^3-3x\right)^3dx \underset{u=x^3-3x\Rightarrow\frac{1}{3}du=\left(x^2-1\right)dx}{=} \frac{1}{3}\int u^3du$$
$$=\tfrac{1}{12}u^4+C=\tfrac{1}{12}\left(x^3-3x\right)^4+C=\tfrac{1}{12}x^{12}-x^{10}+\tfrac{9}{2}x^8-9x^6+\tfrac{27}{4}x^4+C$$

19. Use a u-substitution:

$$y(x)=\int x\sin\left(x^2\right)dx \underset{u=x^2\Rightarrow\frac{1}{2}du=x\,dx}{=} \frac{1}{2}\int\sin u\,du=-\frac{1}{2}\cos u+C=-\frac{1}{2}\cos\left(x^2\right)+C.$$

20. Use a u-substitution:

$$y(x)=\int\frac{x}{\sqrt{x^2-16}}dx \underset{u=x^2-16\Rightarrow\frac{1}{2}du=x\,dx}{=} \frac{1}{2}\int u^{-1/2}du=u^{1/2}+C=\sqrt{x^2-16}+C.$$

21. Use a u-substitution:

$$y(x)=\int\frac{1}{x\ln x}dx \underset{u=\ln x\Rightarrow du=1/x\,dx}{=} =\int\frac{1}{u}du=\ln|u|+C=\ln|\ln x|+C.$$

22. Use integration by parts:

$$y(x)=\int x\ln x\,dx \underset{\substack{u=\ln x\Rightarrow du=1/x\,dx\\ dv=x\,dx\Rightarrow v=x^2/2}}{=} \frac{1}{2}x^2\ln x-\frac{1}{2}\int x^2\bullet\frac{1}{x}dx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C.$$

23. Use integration by parts:

$$y(x)=\int x\,e^{-x}dx \underset{\substack{u=x\Rightarrow du=dx\\ dv=e^{-x}dx\Rightarrow v=-e^{-x}}}{=} -xe^{-x}+\int e^{-x}dx=-xe^{-x}-e^{-x}+C.$$

24. Use partial fractions:

$$y(x)=\int\frac{-2(x+5)}{(x+2)(x-4)}dx=\int\left(\frac{1}{x+2}-\frac{3}{x-4}\right)dx=\ln|x+2|-3\ln|x-4|+C.$$

25. Use partial fractions. First,

$$\frac{x-x^2}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+c}{x^2+1}=\frac{(A+B)x^2+(B+C)x+A+C}{(x+1)\left(x^2+1\right)}.$$

Equating coefficients of like terms yields the system of equations

$$\begin{cases}A+B=-1\\ B+C=1\\ A+C=0\end{cases} \underset{\text{Subtract (2) from (1)}}{\Rightarrow} \begin{cases}A-C=-2\\ A+C=0\end{cases} \underset{\text{Add (1) and (2)}}{\Rightarrow} 2A=-2\Rightarrow A=-1,C=1,B=0.$$

Thus,

$$y(x)=\int\frac{x-x^2}{(x+1)\left(x^2+1\right)}dx=\int\left[-\frac{1}{x+1}+\frac{1}{x^2+1}\right]dx=-\ln|x+1|+\tan^{-1}x+C.$$

26. Use a trigonometric substitution with $x=4\sec\theta$:

$$y(x)=\int\frac{\sqrt{x^2-16}}{x}dx\underset{\substack{x=4\sec\theta\Rightarrow\\dx=4\sec\theta\tan\theta\,d\theta}}{\equiv}\int\frac{\sqrt{(4\sec\theta)^2-16}}{4\sec\theta}4\sec\theta\tan\theta\,d\theta$$

$$=\int\frac{4\tan\theta}{4\sec\theta}4\sec\theta\tan\theta\,d\theta=4\int\tan^2\theta\,d\theta=4\int\left(\sec^2\theta-1\right)d\theta$$

$$=4\tan\theta-4\theta+C\underset{\substack{x=4\sec\theta\Rightarrow\theta=\sec^{-1}\frac{x}{4}\\4\tan\theta=\sqrt{16\sec^2\theta-16}=\sqrt{x^2-16}}}{\equiv}\sqrt{x^2-16}-4\sec^{-1}\frac{x}{4}+C.$$

27. Use a trigonometric substitution with $x=2\sin\theta$ and the identity $\cos^2\theta=\frac{1}{2}(1+\cos2\theta)$ (twice):

$$y(x)=\int\left(4-x^2\right)^{3/2}dx\underset{\substack{x=2\sin\theta\Rightarrow\\dx=2\cos\theta\,d\theta}}{\equiv}\int\left(4-(2\sin\theta)^2\right)^{3/2}\bullet2\cos\theta\,d\theta$$

$$=\int\left(4\cos^2\theta\right)^{3/2}\bullet2\cos\theta\,d\theta=16\int\cos^4\theta\,d\theta$$

$$=16\int\left[\frac{1}{2}(1+\cos2\theta)\right]^2d\theta=4\int\left(1+2\cos2\theta+\cos^2 2\theta\right)d\theta$$

$$=4\int\left[1+2\cos2\theta+\frac{1}{2}(1+\cos4\theta)\right]d\theta=4\int\left(\frac{3}{2}+2\cos2\theta+\frac{1}{2}\cos4\theta\right)d\theta$$

$$=4\left(\frac{3}{2}\theta+\sin2\theta+\frac{1}{8}\sin4\theta\right)+C.$$

Now, $x=2\sin\theta$ so $\theta=\sin^{-1}\left(\frac{x}{2}\right)$. Also, $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\frac{x^2}{4}}=\frac{\sqrt{4-x^2}}{2}$ so

$$\sin2\theta=2\sin\theta\cos\theta=2\bullet\frac{x}{2}\bullet\frac{\sqrt{4-x^2}}{2}=\frac{1}{2}x\sqrt{4-x^2}.$$

Similarly,

$$\cos2\theta=\left\{\begin{array}{c}\cos^2\theta-\sin^2\theta\\2\cos^2\theta-1\\1-2\sin^2\theta\end{array}\right\}=\frac{2-x^2}{2}$$

so

$$\sin4\theta=2\sin2\theta\cos2\theta=2\bullet\frac{1}{2}x\sqrt{4-x^2}\bullet\frac{2-x^2}{2}=\frac{1}{2}x\left(2-x^2\right)\sqrt{4-x^2}.$$

Thus,

$$y(x)=4\left(\frac{3}{2}\theta+\sin2\theta+\frac{1}{8}\sin4\theta\right)+C$$

$$=4\left(\frac{3}{2}\sin^{-1}\left(\frac{x}{2}\right)+\frac{1}{2}x\sqrt{4-x^2}+\frac{1}{8}\bullet\frac{1}{2}x\left(2-x^2\right)\sqrt{4-x^2}\right)+C$$

$$=6\sin^{-1}\left(\frac{x}{2}\right)+2x\sqrt{4-x^2}+\frac{1}{4}x\left(2-x^2\right)\sqrt{4-x^2}+C.$$

28. $\frac{dS}{dt} = -\frac{3(15,000,000)}{(t+100)^4}$, $\frac{dS}{dt} + \frac{3}{t+100}S = -\frac{3(15,000,000)}{(t+100)^4} + \frac{3(15,000,000)}{(t+100)^4} = 0$, $S(0) = 15$, $\lim_{t\to\infty} S(t) = 0$

29. $x(0) = 3$; $\frac{dx}{dt} = -12\sin 4t + 9\cos 4t$, $\frac{dx}{dt}(0) = 9$

30. $u_{xx} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$, $u_{yy} = \frac{x^2 - y^2}{\left(x^2+y^2\right)^2}$, $u_{xx} + u_{yy} = 0$

31. $u_t = 16ke^{-16t}\cos 4x$, $u_{xx} = 16e^{-16t}\cos 4x$; $u(\pi, 0) = 2$; $\lim_{t\to\infty} u(x,t) = 3$

32. $u_{tt} = -\cos t \sin \pi x$, $u_{xx} = -\pi^2 \cos t \sin \pi x$, $u(0,t) = u(1,t) = 0$

33. If $y = x^m$, $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Substitution into the equation yields

$$x^2y'' - 2xy' + 2y = 0$$
$$x^2 \bullet m(m-1)x^{m-2} - 2x \bullet mx^{m-1} + 2x^m = 0$$
$$m(m-1)x^m - 2mx^m + 2x^m = 0$$
$$x^m[m(m-1) - 2m + 2] = 0$$
$$m^2 - 3m + 2 = 0$$
$$(m-2)(m-1) = 0$$

so $m = 1$, $m = 2$.

34. If $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Substitution into the equation yields

$$y'' - 3y' - 18y = 0$$
$$m^2e^{mx} - 3me^{mx} - 18e^{mx} = 0$$
$$e^{mx}\left(m^2 - 3m - 18\right) = 0$$
$$m^2 - 3m - 18 = 0$$
$$(m+3)(m-6) = 0$$

so $m = -3$, $m = 6$.

35.

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^x$$
$$\frac{d}{dx}\left(e^{2x}y\right) = e^x$$
$$e^{2x}y = \int e^x dx = e^x + C$$
$$y = \frac{e^x + C}{e^{2x}} = e^{-x} + Ce^{-2x}$$

36.

$$e^x \frac{dy}{dx} + e^x y = x e^x$$

$$\frac{d}{dx}\left(e^x y\right) = x e^x$$

$$e^x y = \int x e^x dx \underbrace{=}_{\substack{\text{Integration by parts with} \\ u=x \Rightarrow du=dx \\ \text{and} \\ dv=e^x dx \Rightarrow v=e^x}} xe^x - \int e^x dx = xe^x - e^x + C$$

$$y = \frac{xe^x - e^x + C}{e^x} = -1 + x + Ce^{-x}$$

37. Because $\frac{d\psi(x)}{dx} = \frac{An\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$ and $\frac{d^2\psi(x)}{dx^2} = -\frac{An^2\pi^2}{L^2}\sin\left(\frac{n\pi x}{L}\right)$,

$$-\frac{h^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = \frac{h^2}{2m}\frac{An^2\pi^2}{L^2}\sin\left(\frac{n\pi x}{L}\right) = EA\sin\left(\frac{n\pi x}{L}\right).$$

Therefore, $E = \frac{h^2 n^2 \pi^2}{2mL^2}$.

38.

$$-\frac{4}{x^3} + \frac{1}{x^2} + \frac{2}{y^3}\frac{dy}{dx} - \frac{1}{y^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{(x-4)y^3}{x^3(y-2)}; \text{ yes; yes}$$

39. We use implicit differentiation:

$$1 + \frac{2x}{y} - \frac{x^2}{y^2}\frac{dy}{dx} = 0$$

$$-\frac{x^2}{y^2}\frac{dy}{dx} = -1 - \frac{2x}{y}$$

$$\frac{dy}{dx} = \frac{-1 - 2x/y}{-x^2/y^2}$$

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}.$$

Solving $x + \frac{x^2}{y} = C$ for y yields $y = \frac{x^2}{C-x}$. Note that $y = 0$, which is a solution of $\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$, cannot be obtained from $y = \frac{x^2}{C-x}$ so is a singular solution.

40. $x'(t) = 4e^t \sin t - 2e^t \cos t$, $y'(t) = -e^t \cos t - 3e^t \sin t$,
$-2y(t) = 4e^t \sin t - 2e^t \cos t$, $x(t) + 2y(t) = -e^t \cos t - 3e^t \sin t$

41.

$$a\left[C_1 y_1''(x) + C_2 y_2''(x)\right] + b\left[C_1 y_1'(x) + C_2 y_2'(x)\right] + c\left[C_1 y_1(x) + C_2 y_2(x)\right]$$

$$= C_1\left[a y_1''(x) + b y_1'(x) + c y_1(x)\right] + C_2\left[a y_2''(x) + b y_2'(x) + c y_2(x)\right]$$

$$= C_1(0) + C_2(0) = 0$$