ditor: L. Marder

PROBLEM SOLVERS

complex numbers

J. WILLIAMS

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J. WILLIAMS

Senior Lecturer in Applied Mathematics
University of Exeter

LONDON · GEORGE ALLEN & UNWIN LTD

RUSKIN HOUSE MUSEUM STREET

First published in 1972

ISBN 0 04 512018 8 *hardback*
0 04 512019 6 *paper*

Printed in Great Britain
in 10 on 12 pt 'Monophoto' Times Mathematics Series 569
by Page Bros. (Norwich) Ltd., Norwich

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Edited by L. Marder
Senior Lecturer in Mathematics, University of Southampton

No. 6

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Acknowledgements

Thanks are due to Dr J. Herzsberg, Mathematics Department, Birbeck College, University of London, for the care with which he read the manuscript and for making many helpful suggestions; to colleague Miss L. G. Button for her helpful comments; and to Miss J. Knowles for her help towards typing the manuscript.

Contents

Chapter 1

The Algebra of Complex Numbers

1.1 Definitions

1.1 An ordered pair of real numbers x, y, denoted by the symbol $[x, y]$, is termed a *complex number* $z = [x, y]$. The order *is* important, for $[x, y] \neq [y, x]$ unless $x = y$.

1.2 If $z = [x, y]$, we define $-z$ by $[-x, -y]$

1.3 We define the *zero complex number* by $[0, 0]$ and denote it by 0.

1.2 Algebraic Operations We define the following algebraic operations:

Addition. When $z_1 = [x_1, y_1]$ and $z_2 = [x_2, y_2]$,

$$\begin{aligned} z_1 + z_2 &= [x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2] \\ &= [x_2, y_2] + [x_1, y_1] = z_2 + z_1, \end{aligned} \tag{1.1}$$

i.e. the commutative law of addition is satisfied. It follows also that when $z_3 = [x_3, y_3]$,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

i.e. the associative law of addition is satisfied.

Subtraction. $z_1 - z_2 = [x_1, y_1] - [x_2, y_2] = [x_1 - x_2, y_1 - y_2]$ (1.2)

from which it follows that $z_1 - z_2 = z_1 + (-z_2)$ using Definition (1.2).

Equality. $z_1 = z_2$, when $z_1 - z_2 = 0$, i.e. using (1.2) with Definition (1.3),

$$[x_1, y_1] = [x_2, y_2] \Leftrightarrow x_1 = x_2 \quad \text{and} \quad y_1 = y_2. \tag{1.3}$$

Multiplication. (i) A real number a, with a complex number:

$$a[x, y] = [ax, ay]$$

(ii) Two complex numbers:

$$\begin{aligned} z_1 z_2 &= [x_1, y_1] \times [x_2, y_2] = [x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2] \\ &= [x_2, y_2] \times [x_1, y_1] = z_2 z_1 \end{aligned} \tag{1.4}$$

so that multiplication follows the commutative law. Furthermore, $(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3)$ (see Problem 1.1), so that the associative law also applies to the operation of multiplication. Also

$$(z_1 + z_2) \times z_3 = z_1 \times z_3 + z_2 \times z_3$$

(The proof of this is left to the reader) so that the distributive law is satisfied.

Note: For the product of z_1 and z_2 we use $z_1 \times z_2$ or $z_1 \cdot z_2$ or even $z_1 z_2$.

Division. If $z_2 \neq 0$ we define $z_1 \div z_2$ (also written z_1/z_2) as the complex number $[\alpha, \beta]$, i.e. $[x_1, y_1] \div [x_2, y_2]$ is defined as the complex number $[\alpha, \beta]$ where $[\alpha, \beta] \times [x_2, y_2] = [x_1, y_1]$. Using (1.4), the equations which determine α, β (uniquely) are

$$\alpha x_2 - \beta y_2 = x_1 \quad \text{and} \quad \alpha y_2 + \beta x_2 = y_1,$$

whence
$$\alpha = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \qquad \beta = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

unless x_2 and y_2 are both zero. Consequently, provided $[x_2, y_2] \neq [0, 0]$, the quotient z_1/z_2 is uniquely determined by

$$\frac{z_1}{z_2} = \frac{[x_1, y_1]}{[x_2, y_2]} = \left[\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right]. \tag{1.5}$$

Problem 1.1 Prove the associative law of multiplication,

$$(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3).$$

Solution. Using (1.4), $z_1 z_2 = [X, Y]$, where $X = x_1 x_2 - y_1 y_2$, $Y = x_1 y_2 + y_1 x_2$; and $(z_1 z_2) z_3 = [L, M] = [X, Y][x_3, y_3]$. Then

$$L = X x_3 - Y y_3 = x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3$$
$$= X' x_1 - Y' y_1$$

where $\quad X' = x_2 x_3 - y_2 y_3, \qquad Y' = y_2 x_3 + x_2 y_3$

$$M = X y_3 + Y x_3 = x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3$$
$$= X' y_1 + Y' x_1$$

so that $[L, M] = [x_1, y_1][X', Y']$

But $\quad [X', Y'] = [x_2 x_3 - y_2 y_3, y_2 x_3 + x_2 y_3] = [x_2, y_2][x_3, y_3]$ □

from which the result follows.

Problem 1.2 Evaluate (a) $[3, -2] + [2, -3]$, (b) $[3, -2] - [2, -3]$, (c) $[3, -2]^2 - [2, -3]^2$.

Solution.

(a) $[3, -2] + [2, -3] = [3+2, -2+(-3)] = [5, -5]$

(b) $[3, -2] - [2, -3] = [3-2, -2-(-3)] = [1, 1]$

(c) $[3, -2]^2 = [3, -2] \times [3, -2]$
$= [3.3 - (-2).(-2), 3(-2) + (-2).3] = [5, -12]$

$[2, -3]^2 = [2, -3] \times [2, -3]$
$= [2.2 - (-3).(-3), 2(-3) + (-3).[-5, -12].$

Hence $[3, -2]^2 - [2, -3]^2 = [5, -12] - [-5, -12]$
$$= [5-(-5), -12-(-12)] = [10, 0].$$

Alternatively, factorizing and using (a) and (b),

$$\begin{aligned}[3, -2]^2 - [2, -3]^2 &= \{[3, -2]+[2, -3]\} \times \{[3, -2]-[2, -3]\} \\ &= [5, -5] \times [1, 1] \\ &= [5.1-(-5).1, 5.1+(-5).1] = [10, 0].\end{aligned}$$ □

Problem 1.3 Evaluate (a) $[\cos\theta, \sin\theta] \times [\cos\phi, \sin\phi]$, (b) $[\cos\theta, \sin\theta] \div [\cos\phi, \sin\phi]$, (c) $[\cos\theta, \sin\theta]^n$, where n is a positive integer.

Solution.

(a) $[\cos\theta, \sin\theta] \times [\cos\phi, \sin\phi]$

$$\begin{aligned} &= [\cos\theta\cos\phi - \sin\theta\sin\phi, \sin\theta\cos\phi + \cos\theta\sin\phi] \\ &= [\cos(\theta+\phi), \sin(\theta+\phi)].\end{aligned}$$

(b) $[\cos\theta, \sin\theta] \div [\cos\phi, \sin\phi] = [\alpha, \beta]$ where $\cos\theta = \alpha\cos\phi - \beta\sin\phi$, $\sin\theta = \alpha\sin\phi + \beta\cos\phi$. Solving for α and β we have

$$\begin{aligned}\alpha &= \cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta-\phi) \\ \beta &= \sin\theta\cos\phi - \cos\theta\sin\phi = \sin(\theta-\phi)\end{aligned}$$

so that $\quad [\cos\theta, \sin\theta]/[\cos\phi, \sin\phi] = [\cos(\theta-\phi), \sin(\theta-\phi)].$

If we multiply both sides by the denominator of the left-hand side, (which is not $[0, 0]$ since $\cos\phi$, $\sin\phi$ cannot be simultaneously zero), we have, using (a),

$$\begin{aligned}[\cos\theta, \sin\theta] &= [\cos\phi, \sin\phi]\,[\cos(\theta-\phi), \sin(\theta-\phi) \\ &= [\cos(\phi+\theta-\phi), \sin(\phi+\theta-\phi)] = [\cos\theta, \sin\theta].\end{aligned}$$

(c) When $n = 1$, the result is obviously $[\cos\theta, \sin\theta]$.
For $n = 2$, using (a),

$$[\cos\theta, \sin\theta]^2 = [\cos\theta, \sin\theta]\,[\cos\theta, \sin\theta] = [\cos 2\theta, \sin 2\theta].$$

For $n = 3$,

$$\begin{aligned}[\cos\theta, \sin\theta]^3 &= [\cos\theta, \sin\theta]^2\,[\cos\theta, \sin\theta] \\ &= [\cos 2\theta, \sin 2\theta]\,[\cos\theta, \sin\theta] = [\cos 3\theta, \sin 3\theta].\end{aligned}$$

These results suggest a general result of the form $[\cos\theta, \sin\theta]^n = [\cos n\theta, \sin n\theta]$; it is certainly true for $n = 1, 2$ and 3. If it is true for *some* positive integral value n, this formula is true for $n+1$, for using (a) again,

$$\begin{aligned}[\cos\theta, \sin\theta]^{n+1} &= [\cos\theta, \sin\theta]^n[\cos\theta, \sin\theta] \\ &= [\cos n\theta, \sin n\theta]\,[\cos\theta, \sin\theta] \\ &= [\cos(n+1)\theta, \sin(n+1)\theta],\end{aligned}$$

i.e. the result if true for n is also true for $n+1$ and since it is true for $n = 3$, it is true for $n = 4$ and so on for any positive integer n. Consequently, when n is a positive integer,

$$[\cos\theta, \sin\theta]^n = [\cos n\theta, \sin n\theta]. \qquad \square$$

1.3 The Imaginary Quantity *i* Suppose that we consider the complex numbers $z_r = [x_r, 0]$, $r = 1, 2$ then $z_1 \pm z_2 = [x_1 \pm x_2, 0]$, $z_1 z_2 = [x_1, x_2, 0]$ and if $x_2 \neq 0$, $z_1/z_2 = [x_1/x_2, 0]$, i.e. these numbers behave in a similar way to real numbers under the ordinary algebraic operations. Hence we can identify the complex number $[x, 0]$ with the real number x. Moreover, $[0, y]^2 = [0, y][0, y] = [-y^2, 0] = -y^2$ a negative real number i.e. $[0, y]$ cannot be identified with any real number. This suggests that we write the complex number $z = [x, y]$ in the form $z = x+iy$ where i is a non-real number and is therefore, not equatable to any real number. This implies that

$$x_1+iy_1 = x_2+iy_2 \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2.$$

We stipulate that the quantities x_1, x_2, iy_1, iy_2 when added or multiplied obey the seven fundamental laws of algebra so that

$$(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2),$$

or

$$[x_1, y_1]+[x_2, y_2] = [x_1+x_2, y_1+y_2].$$

Again, $(x_1+iy_1)(x_2+iy_2) = x_1x_2+y_1 y_2 i^2+(x_1 y_2+y_1 x_2)i$. If we stipulate that $i^2 = -1$, the right-hand side representing the product is

$$x_1 x_2-y_1 y_2+(x_1 y_2+y_1 x_2)i,$$

so that we retrieve the multiplication rule

$$[x_1, y_1][x_2, y_2] = [x_1 x_2-y_1 y_2, x_1y_2+y_1 x_2].$$

Since, when x is a real number, x^2 is *never* negative, it follows that i is a non-real or *imaginary* number. Furthermore, both $z = i$ and $z = -i$ are roots of the equation $z^2 = -1$.

Definition 1.4 When $z = x+iy$, we refer to as the *real part* of z, whereas y (which is a *real* number) is called the *imaginary part* of this complex number z.

We write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. Thus when $z = 2+i$, $\operatorname{Re}(2+i) = 2$, $\operatorname{Im}(2+i) = 1$. z is purely real if $y = 0$ and purely imaginary if $x = 0$.

1.4 The Complex Conjugate

Definition 1.5 Given any complex number $z = x+iy$, the *conjugate* complex number written as $\bar{z}$, (or sometimes z^*) is defined by $\bar{z} = x-iy$, i.e. to find the conjugate, simply change the sign of the imaginary part, e.g. when $z = -2+3i$, $\bar{z} = -2-3i$.

Formulae involving conjugates are:

(1.6) (i) $\bar{\bar{z}} = z$ or $z^{**} = z$, (ii) $\bar{z} = z \Leftrightarrow \operatorname{Im} z = 0$

(iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (iv) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(v) $\overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2$ (vi) $\operatorname{Re} z = \operatorname{Re} \bar{z} = \frac{1}{2}(z+\bar{z})$

(vii) $\operatorname{Im} \bar{z} = -\operatorname{Im} z = \frac{1}{2}(z-\bar{z})i$.

The proofs of (i) to (vi) are elementary and are left as an exercise for the reader.

Two important quantities which can be used to define complex numbers in place of the real and imaginary parts are the *modulus* and *argument*.

Definition 1.6 If in $z = x+iy$ we write $x = r\cos\theta$, $y = r\sin\theta$, where r is the non-negative value of $\sqrt{(x^2+y^2)}$, this quantity r is called the *modulus* of z. We write

$$r = |z| = \operatorname{mod} z, \qquad z = r\cos\theta + ir\sin\theta,$$

(Note that $r = 0 \Leftrightarrow x = 0 = y$).

Definition 1.7 Given x and y, not both zero, and $z = x+iy$, *any* value of θ (measured in radians) satisfying both $\cos\theta = x/r$, $\sin\theta = y/r$, is called an *argument* of z, and is written

$$\theta = \operatorname{Arg} z, \qquad \text{(with } \textit{capital A}\text{)}.$$

The one value of θ which lies in the interval $-\pi < \theta \leqslant \pi$ is defined to be the *principal value* of the argument and is written

$$\theta = \arg z, \qquad \text{(with } \textit{small a}\text{)}.$$

It is obvious that $\operatorname{Arg} z = \arg z + 2n\pi$ where n is some positive or negative integer. When n is known $\operatorname{Arg} z$ could be written $\arg_n z$. It follows that the principal value of the argument of a real number is 0 if the number is positive and π if the number is negative. The argument of zero is undefined.

Formulae involving moduli and arguments are:

(1.7) (i) $|z_1 z_2| = |z_1||z_2|$.

(ii) $|z_1/z_2| = |z_1|/|z_2|$, $z_2 \neq 0$.

(iii) $|z^n| = |z|^n$, n integral.

(iv) $|z|^2 = |\bar{z}|^2 = z\bar{z} = r^2 = x^2+y^2$.

(v) $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$, $z_1 \neq 0$, $z_2 \neq 0$, i.e. *any* value of $\operatorname{Arg} z_2$ gives *a* value of $\operatorname{Arg}(z_1 z_2)$ with a similar interpretation.

(vi) $\operatorname{Arg}(z_1/z_2) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$, $z_1 \neq 0$, $z_2 \neq 0$.

(vii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2\delta\pi$, $z_1 \neq 0, z_2 \neq 0$ where $\delta = -1, 0, 1$ according as $\pi < \phi \leqslant 2\pi$, $-\pi < \phi \leqslant \pi$ or $-2\pi < \phi \leqslant -\pi$ where $\phi = \arg z_1 + \arg z_2$.

(viii) $\arg(z_1/z_2) = \arg z_1 - \arg z_2 + 2\delta\pi$, $z_1 \neq 0$, $z_2 \neq 0$, where $\delta = -1, 0, 1$ according as $\arg z_1 - \arg z_2 = \phi$ satisfies $\pi < \phi < 2\pi$, $-\pi < \phi < \pi$, or $-2\pi < \phi < -\pi$.

(ix) $\arg z^n = n \arg z + 2p\pi$, $z \neq 0$, n integral, where p is an integer satisfying $-\pi < n \arg z + 2p\pi \leqslant \pi$.

Except for (i) and (v) we leave the proofs to the reader. To prove (i) and (v) we write

$$z_1 = r_1(\cos\theta + i\sin\theta), \qquad z_2 = r_2(\cos\phi + i\sin\phi).$$

Using Problem 1.3(a) we have

$$z_1 z_2 = r_1 r_2\{\cos(\theta+\phi) + i\sin(\theta+\phi)\},$$

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2| \quad \text{and} \quad \operatorname{Arg}(z_1 z_2) = \theta + \phi = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

1.5 Inequalities Whereas real numbers form an *ordered* system in which we can write $a >, =,$ or $< b$ when a and b are real, this property does not extend to complex numbers, i.e. we *do not write* $z_1 > z_2$, for it has *no meaning*. However, the real numbers $\operatorname{Re} z$, $\operatorname{Im} z$ or $|z|$ formed from a complex z *do* possess the property of order. Formulae involving inequalities are:

(1.8) (i) $\operatorname{Re} z \leqslant |\operatorname{Re} z| \leqslant |z|$.

(ii) $\operatorname{Im} z \leqslant |\operatorname{Im} z| \leqslant |z|$.

(iii) $|z_1 + z_2| \leqslant |z_1| + |z_2|$, the *triangle inequality*.

(iv) $|z_1| + |z_2| \geqslant |z_1 - z_2| \geqslant |z_1| - |z_2|$.

(v) $\left|\sum_{n=1}^{N} z_n\right| \leqslant \sum_{n=1}^{N} |z_n|$, the extension of (iii).

(vi) $|z_n - z_1| \leqslant \sum_{m=1}^{n-1} |z_{m+1} - z_m|$.

Proof of (i), (ii) follows from $|z| = (\operatorname{Re}^2 z + \operatorname{Im}^2 z)^{\frac{1}{2}}$. To prove (iii), we have

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1 \\ &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) \\ &\leqslant |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2, \end{aligned}$$

i.e. $|z_1 + z_2| \leqslant |z_1| + |z_2|$. Again, the proofs of the others are left to the reader.

Problem 1.4 Simplify (i) $(7+i)(4-i)$, (ii) $(7+i)/(4-i)$, (iii) $[(1+2i)/(2+i)]^3$.

Solution

(*i*) $(7+i)(4-i) = 28+4i-7i-i^2 = 28-3i+1 = 29-3i.$

(ii)
$$\frac{7+i}{4-i} = \left[\frac{7+i}{4-i}\right]\left[\frac{4+i}{4+i}\right] = \frac{28+4i+7i+i^2}{16-i^2} = \frac{28+11i-1}{17}$$
$$= \frac{27+11i}{17}.$$

(iii)
$$\left[\frac{1+2i}{2+i}\right]^3 = \frac{1+3(2i)+3(2i)^2+(2i)^3}{2^3+3.2^2i+3.2i^2+i^3} = \frac{1+6i-12-8i}{8+12i-6-i}$$
$$= -\left[\frac{11+2i}{2+11i}\right] = -\left[\frac{11+2i}{2+11i}\right]\left[\frac{2-11i}{2-11i}\right]$$
$$= -\frac{22+4i-121i+22}{4+121} = \frac{-44+117i}{125}$$

Alternatively,
$$\frac{1+2i}{2+i} = \frac{1+2i}{2+i}\left[\frac{2-i}{2-i}\right] = \frac{2+4i-i+2}{4+i} = \frac{4+3i}{5},$$

so that
$$\left[\frac{1+2i}{2+i}\right]^3 = \left[\frac{4+3i}{5}\right]^3 = \frac{4^3+3.4^2(3i)+34(3i)^2+(3i)^3}{125}$$
$$= \frac{64+144i-108-27i}{125} = \frac{-44+117i}{125}, \text{ as before.}$$

Problem 1.5 Express $(1+i)(1+i\sqrt{3})(\sqrt{3}-i)$ in the form $a+ib$ (a,b both real) and find its modulus and argument.

Solution.
$$a+ib = (1+i)(1+i\sqrt{3})(\sqrt{3}-1) = (1+i)(\sqrt{3}+3i-i+\sqrt{3})$$
$$= 2(1+i)(\sqrt{3}+i) = 2\{\sqrt{3}-1+i(\sqrt{3}+1)\},$$

i.e. $a = 2(\sqrt{3}-1)$, $b = 2(\sqrt{3}+1)$. If $r = \text{mod}(a+ib)$, $\theta = \arg(a+ib)$, where $-\pi < \theta \leqslant \pi$, then $r\cos\theta = a = 2(\sqrt{3}-1)$, $r\sin\theta = b = 2(\sqrt{3}+1)$, and $r^2 = a^2+b^2 = 4\{(\sqrt{3}-1)^2+(\sqrt{3}+1)^2\} = 8\{(\sqrt{3})^2+1\} = 32.$

Hence $r = 4\sqrt{2} = |a+ib|$, $\cos\theta = \frac{1}{4}\sqrt{2}(\sqrt{3}-1)$, $\sin\theta = \frac{1}{4}\sqrt{2}(\sqrt{3}+1)$. At this stage we can either refer to tables of the trigonometrical functions to obtain the result $\theta = 5\pi/12$ or using $\sin 2\theta = 2\cos\theta\sin\theta = \frac{1}{2}$ we deduce that $2\theta = \frac{1}{6}\pi$ or $\frac{5}{6}\pi$.

Now, to decide which root gives the correct value for θ, we evaluate $\cos 2\theta$ using
$$\cos 2\theta = 2\cos^2\theta-1 = \tfrac{1}{2}(2-\sqrt{3})-1 = -\tfrac{1}{2}\sqrt{3},$$

i.e. $2\theta = \frac{5}{6}\pi$ (and *not* $\frac{1}{6}\pi$ which would make $\cos 2\theta$ positive). Hence

$$\arg(a+ib) = 5\pi/12.$$

Alternatively, to find the modulus and argument we could exploit the duplication formulae (1.7) (i) and (v) as follows.

$$z_1 = 1+i = r_1(\cos\theta_1 + i\sin\theta_1);$$

$$r_1\cos\theta_1 = 1, r_1\sin\theta_1 = 1; \quad r = \sqrt{2}, \theta_1 = \tfrac{1}{4}\pi.$$

$$z_2 = 1+\sqrt{3}i = r_2(\cos\theta_2 + \sin\theta_2);$$

$$r_2\cos\theta_2 = 1, r_2\sin\theta_2 = \sqrt{3}; \quad r = 2, \theta_2 = \tfrac{1}{3}\pi.$$

$$z_2 = \sqrt{3}-i = r_3(\cos\theta_3 + i\sin\theta_3); \quad r_3\cos\theta_3 = \sqrt{3}, r_3\sin\theta_3 = -1;$$

$$r_3 = 2, \theta_3 = -\tfrac{1}{6}\pi.$$

As before,

$$|z_1 z_2 z_3| = r_1 r_2 r_3 = 4\sqrt{2}$$

$$\text{Arg}(z_1 z_2 z_3) = \theta_1+\theta_2+\theta_3 = \tfrac{1}{4}\pi+\tfrac{1}{3}\pi-\tfrac{1}{6}\pi = 5\pi/12,$$

which turns out to be the principal value of the argument since it satisfies the inequality $-\pi < \theta \leqslant \pi$. □

Problem 1.6 Find the modulus and argument of (i) $1+\cos\theta+i\sin\theta$, (ii) $1-\cos\theta+i\sin\theta$, and deduce the corresponding values for (iii) $(1-i)(1+\cos\theta+i\sin\theta)/(1+i)(1-\cos\theta+i\sin\theta)$ when $-2\pi < \theta < 2\pi$, $\theta \neq 0$ or $\pm\pi$.

Solution. (i) $1+\cos\theta+i\sin\theta = 2\cos\frac{1}{2}\theta(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta)$ which is of the form $r(\cos\phi+i\sin\phi)$, i.e. the modulus $r = 2|\cos\frac{1}{2}\theta|$; it should be remembered that r must take the *positive* value. Let us assume that $-2\pi \leqslant \theta \leqslant 2\pi$; other values of θ will yield corresponding results in r and ϕ.

When $\theta = \pm\pi$, $r = 0$ and ϕ, the argument is not defined.

When $-\pi < \theta < \pi$, $r = 2\cos\frac{1}{2}\theta$, since the cosine is positive, and $\phi = \frac{1}{2}\theta$ *is* a principal value because $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$.

When $\pi < \theta < 2\pi$, or $-2\pi < \theta < -\pi$, $\cos\frac{1}{2}\theta$ is negative so that here $r = -2\cos\frac{1}{2}\theta$, $\cos\phi = -\cos\frac{1}{2}\theta$, $\sin\phi = -\sin\frac{1}{2}\theta$, i.e. $\phi = \pi+\frac{1}{2}\theta$ or $\frac{1}{2}\theta-\pi$.

For $\pi < \theta < 2\pi$, $-\frac{1}{2}\pi < \phi < 0$ when $\phi = \frac{1}{2}\theta-\pi$, i.e. ϕ *is* a principal value and for $-2\pi < \theta < -\pi$, $0 < \phi < \frac{1}{2}\pi$ when $\phi = \frac{1}{2}\theta+\pi$, i.e. ϕ *is* a principal value.

(ii) We can, of course, deduce the results of this case from (i) by substituting $\pi-\theta$ for θ, or we have

$$1-\cos\theta+i\sin\theta = 2\sin\tfrac{1}{2}\theta(\sin\tfrac{1}{2}\theta+i\cos\tfrac{1}{2}\theta), \quad \text{i.e. } r = 2|\sin\tfrac{1}{2}\theta|.$$

When $\theta = 0, \pm 2\pi$, $r = 0$ and the argument ψ is not defined.

When $0 < \theta < 2\pi$, $r = 2\sin\frac{1}{2}\theta > 0$, $\cos\psi = \sin\frac{1}{2}\theta$, $\sin\psi = \cos\frac{1}{2}\theta$, so that $\psi = \frac{1}{2}(\pi-\theta)$ with $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$ is the principal valued argument.

When $-2\pi < \theta < 0$, $\sin\frac{1}{2}\theta$ is negative so that $r = -2\sin\frac{1}{2}\theta$, $\cos\psi = -\sin\frac{1}{2}\theta$, $\sin\psi = -\cos\frac{1}{2}\theta$, i.e. $\psi = -\frac{1}{2}(\pi+\theta)$, $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$, which gives ψ its principal value.

(iii) $(1-i) = \sqrt{2}[\cos(-\frac{1}{4}\pi)+i\sin(-\frac{1}{4}\pi)]$ and $(1+i) = \sqrt{2}[\cos(\frac{1}{4}\pi)+i\sin(\frac{1}{4}\pi)]$. Consequently, the modulus R of the expression using (i) and (ii) of (1.7) and the above is

$$R = 2\sqrt{2}|\cos\tfrac{1}{2}\theta|/2\sqrt{2}|\sin\tfrac{1}{2}\theta| = |\cot\tfrac{1}{2}\theta|.$$

We now exclude the points $\theta = \pm\pi$ at which the expression is zero and the points $\theta = 0, \pm 2\pi$ at which the denominator is zero leaving the quotient undefined or *singular*; these facts were, of course, anticipated in the original statement of the problem.

For $-2\pi < \theta < -\pi$ and $0 < \theta < \pi$, $\cot\frac{1}{2}\theta$ is positive so that $R = \cot\frac{1}{2}\theta$. But for $-\pi < \theta < 0$ and $\pi < \theta < 2\pi$, $\cot\frac{1}{2}\theta$ is negative and $R = -\cot\frac{1}{2}\theta$.

To determine the argument of the quotient we use (1.7) (v) and (vi), i.e. ω the argument, which will be the principal value only if $-\pi < \omega \leqslant \pi$, is given by

$$\begin{aligned}\omega &= \arg(1-i)-\arg(1+i)+\arg(1+\cos\theta+i\sin\theta)-\arg(1-\cos\theta+i\sin\theta)\\ &= -\tfrac{1}{4}\pi-\tfrac{1}{4}\pi+\phi-\psi.\end{aligned}$$

This argument will vary with different intervals of θ. Using the results of (ii) and (iii) we have for $-2\pi < \theta < -\pi$, $\phi = \frac{1}{2}\theta+\pi$, $\psi = -\frac{1}{2}(\pi+\theta)$, i.e. $\omega = \theta+\pi$. When $-\pi < \theta < 0$, $\phi = \frac{1}{2}\theta$, $\psi = -\frac{1}{2}(\pi+\theta)$, i.e. $\omega = \theta$. When $0 < \theta < \pi$, $\phi = \frac{1}{2}\theta$, $\psi = \frac{1}{2}(\pi-\theta)$, i.e. $\omega = \theta-\pi$. Finally when $\pi < \theta < 2\pi$, $\phi = \frac{1}{2}\theta-\pi$, $\psi = \frac{1}{2}(\pi-\theta)$, i.e. $\omega = \theta-2\pi$. In each case $-\pi < \omega < 0$, so that ω is the principal value. □

Problem 1.7 Find the modulus and the principal value of the argument of Z where

$$Z = \cos\alpha - i\sin\alpha+\cos\theta+i\sin\theta \qquad (0 < \alpha < \tfrac{1}{2}\pi).$$

Solution. Adding the components of the real and imaginary parts separately, we have

$$\begin{aligned}&2\cos\tfrac{1}{2}(\theta+\alpha)\cos\tfrac{1}{2}(\theta-\alpha)+2i\sin\tfrac{1}{2}(\theta-\alpha)\cos\tfrac{1}{2}(\theta+\alpha)\\ &\quad= 2\cos\tfrac{1}{2}(\theta+\alpha)\{\cos\tfrac{1}{2}(\theta-\alpha)+i\sin\tfrac{1}{2}(\theta-\alpha)\}.\end{aligned}$$

The modulus is $r = |2\cos\frac{1}{2}(\theta+\alpha)|$. To determine the argument we discuss separately the cases in which $\cos\frac{1}{2}(\theta+\alpha)$ is positive or negative. We exclude the values $\theta = \pm\pi-\alpha$ at which $r = 0$ and $Z = 0$.

Case (*i*). $\cos\frac{1}{2}(\theta+\alpha) > 0$ when $-\frac{1}{2}\pi < \frac{1}{2}(\theta+\alpha) < \frac{1}{2}\pi$. Here

$$r = |Z| = 2\cos\tfrac{1}{2}(\theta+\alpha), \qquad \operatorname{Arg} Z = \tfrac{1}{2}(\theta-\alpha)+2k\pi \quad (k \text{ integral}).$$

Case (ii). $\cos\frac{1}{2}(\theta+\alpha) < 0$ when $\frac{1}{2}\pi < \frac{1}{2}(\theta+\alpha) < \frac{3}{2}\pi$. Here

$$r = |Z| = -2\cos\tfrac{1}{2}(\theta+\alpha), \qquad \operatorname{Arg} Z = \pi+\tfrac{1}{2}(\theta-\alpha)+2k\pi, \quad (k \text{ integral}).$$

In case (i), since $-\frac{1}{2}\pi < \frac{1}{2}(\theta+\alpha) < \frac{1}{2}\pi$ we have $-\frac{1}{2}\pi-\alpha < \frac{1}{2}(\theta-\alpha) < \frac{1}{2}\pi-\alpha$. Hence when $0 < \alpha < \frac{1}{2}\pi$, $\frac{1}{2}(\theta-\alpha)$ lies in the open interval $(-\pi, \frac{1}{2}\pi)$ and $\arg Z = \frac{1}{2}(\theta-\alpha)$.

In case (ii) where $\frac{1}{2}\pi < \frac{1}{2}(\theta+\alpha) < \frac{3}{2}\pi$, we have $\frac{1}{2}\pi-\alpha < \frac{1}{2}(\theta-\alpha) < \frac{3}{2}\pi-\alpha$. When $0 < \alpha < \frac{1}{2}\pi$, $\frac{1}{2}(\theta-\alpha)$ lies in the open interval $(0, \frac{3}{2}\pi)$, and $\operatorname{Arg} Z = \pi+\frac{1}{2}(\theta-\alpha)$ with $k = 0$ lies in $(\pi, \frac{5}{2}\pi)$.

Hence, to obtain $\arg z$, the principal value, we must choose $k = -1$ in the general formula $\operatorname{Arg} Z = \pi+\frac{1}{2}(\theta-\alpha)+2k\pi$, i.e., $\arg Z = \frac{1}{2}(\theta-\alpha)-\pi$ which lies in $(-\pi, \frac{1}{2}\pi)$. □

Problem 1.8 Given $|z+16| = 4|z+1|$, deduce that $|z| = 4$.

Solution. We have

$$|z+16|^2 = (z+16)(\bar{z}+16) = z\bar{z}+16(z+\bar{z})+256$$

$$|z+1|^2 = (z+1)(\bar{z}+1) = z\bar{z}+z+\bar{z}+1.$$

But $|z+16|^2 = 16|z+1|^2$, i.e.

$$z\bar{z}+16(z+\bar{z})+256 = 16(z\bar{z}+z+\bar{z}+1)$$

or

$$240 = 15z\bar{z}.$$

Hence $z\bar{z} = |z|^2 = 240/15 = 16$. Taking the square root, $|z| = 4$. □

Problem 1.9 Given $z^{-1} = (a+ib)^{-1}+(a+ic)^{-1}$, $z = x+iy$, a, b, c being real, with $a+ib$, $a+ic$ not zero, evaluate (i) x^2+y^2, (ii) $(x-a)^2+y$. Hence deduce $\operatorname{Re} z$.

Solution.

$$\frac{1}{z} = \frac{2a+i(b+c)}{(a+ib)(a+ic)}, \qquad \text{i.e. } z = \frac{(a+ib)(a+ic)}{2a+i(b+c)},$$

so that

$$x^2+y^2 = z\bar{z} = \frac{(a+ib)(a+ic)}{2a+i(b+c)}\cdot\frac{(a-ib)(a-ic)}{2a-i(b+c)} = \frac{(a^2+b^2)(a^2+c^2)}{4a^2+(b+c)^2}. \quad \text{(i)}$$

To evaluate (ii) we consider $z-a$, which is $(x-a)+iy$.

$$z-a = \frac{(a+ib)(a+ic)}{2a+i(b+c)}-a = \frac{a^2+ib(b+c)-bc-a\{2a+i(b+c)\}}{2a+i(b+c)},$$

i.e.

$$(x-a)+iy = -(a^2+bc)/\{2a+i(b+c)\}.$$

Taking the conjugate of both sides,

$$(x-a)-iy = -(a^2+bc)/\{2a-i(b+c)\},$$

so that multiplying, we have

$$(x-a+iy)(x-a-iy) = (x-a)^2+y^2 = (a^2+bc)^2/\{4a^2+(b+c)^2\}. \quad \text{(ii)}$$