Introduction to
Linear Algebra 2nd ed

INTRODUCTION TO LINEAR ALGEBRA

Second Edition

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PREFACE

This is a basic textbook for linear algebra, combining the theory with the applications. The central equations are the linear system $Ax = b$ and the eigenvalue problem $Ax = \lambda x$. It is simply amazing how much there is to say (and learn) about those two equations. This book comes from years of teaching and organizing and thinking about the linear algebra course—and still this subject feels new and very alive.

I am really happy that the need for linear algebra is widely recognized. It is *absolutely* as important as calculus. I don't concede anything, when I look at how mathematics is actually used. So many applications today are discrete rather than continuous, digital rather than analog, linearizable rather than erratic and chaotic. Then vectors and matrices are the language to know.

The equation $Ax = b$ uses that language right away. The left side has a matrix A times an unknown vector x. ***Their product Ax is a combination of the columns of A.*** This is the best way to multiply, and the equation is asking for the combination that produces b. Our solution can come at three levels and they are all important :

1. ***Direct solution*** by forward elimination and back substitution.
2. ***Matrix solution*** by $x = A^{-1}b$ using the inverse matrix A^{-1}.
3. ***Vector space solution*** by finding all combinations of the columns of A and all solutions to $Ax = 0$. We are looking at the column space and the nullspace.

And there is another possibility: $Ax = b$ *may have no solution*. The direct approach by elimination may lead to $0 = 1$. The matrix approach may fail to find A^{-1}. The vector space approach can look at all combinations of the columns, but b might not lie in that column space. Part of the mathematics is understanding when an equation is solvable and when it's not.

Another part is learning to visualize vectors. A vector v with two components is not hard. The components v_1 and v_2 tell how far to go across and up—we can draw an arrow. A second vector w may be perpendicular to v (and Chapter 1 tells exactly when). If those vectors have six components, we can't draw them but our imagination keeps trying. We can think of a right angle in six-dimensional space. We can see $2v$ (twice as far) and $-w$ (in the opposite direction to w). We can *almost* see a combination like $2v - w$.

Most important is the effort to imagine ***all the combinations of cv with dw***. They fill some kind of "two-dimensional plane" inside the six-dimensional space. As I write these words, I am not at all sure that I see this subspace. But linear algebra offers a simple way to work with vectors and matrices of any size. If we have six currents in a network or six forces on a structure or six prices for our products, we are certainly in six dimensions. In linear algebra, a six-dimensional space is pretty small.

Already in this preface, you can see the style of the book and its goal. The style is informal but the goal is absolutely serious. Linear algebra is great mathematics, and I try to explain it as clearly as I can. I certainly hope that the professor in this course learns new things. The author always does. The student will notice that the applications reinforce the ideas. That is the whole point for all of us—to learn how to think. I hope you will see how this book moves forward, *gradually but steadily*.

Mathematics is continually asking you to look beyond the most specific case, to see the broader pattern. Whether we have pixel intensities on a TV screen or forces on an airplane or flight schedules for the pilots, those are all vectors and they all get multiplied by matrices. Linear algebra is worth doing well.

Structure of the Textbook

I want to note five points about the organization of the book:

1. Chapter 1 provides a brief introduction to the basic ideas of vectors and matrices and dot products. If the class has met them before, there is no problem to begin with Chapter 2. That chapter solves n by n systems $Ax = b$.

2. For rectangular matrices, I now use the *reduced row echelon form* more than before. In MATLAB this is $R = \mathbf{rref}(A)$. Reducing A to R produces bases for the row space and column space. Better than that, reducing the combined matrix $[\,A \;\; I\,]$ produces total information about all four of the fundamental subspaces.

3. Those four subspaces are an excellent way to learn about linear independence and dimension and bases. The examples are so natural, and they are genuinely the key to applications. I hate just making up vector spaces when so many important ones are needed. If the class sees plenty of examples of independence and dependence, then the definition is virtually understood in advance. The columns of A are independent when $x = \mathbf{0}$ is the only solution to $Ax = \mathbf{0}$.

4. Section 6.1 introduces eigenvalues for ***2 by 2 matrices.*** Many courses want to meet eigenvalues early (to apply them in another subject or to avoid missing them completely). It is absolutely possible to go directly from Chapter 3 to Section 6.1. The determinant is easy for a 2 by 2 matrix, and eigenvalues come through clearly.

5. Every section in Chapters 1 to 7 ends with a highlighted ***Review of the Key Ideas***. The reader can recapture the main points by going carefully through this review.

A one-semester course that moves steadily can reach eigenvalues. The key idea is to *diagonalize a matrix*. For most square matrices that is $S^{-1}AS$, using the eigenvector matrix S. For symmetric matrices it is Q^TAQ. When A is rectangular we need U^TAV. I do my best to explain that Singular Value Decomposition because it has become extremely useful. I feel very good about this course and the student response to it.

Structure of the Course

Chapters 1-6 are the heart of a basic course in linear algebra—*theory plus applications*. The beauty of this subject is in the way those two parts come together. The theory is needed, and the applications are everywhere.

I now use the web page to post the syllabus and homeworks and exam solutions:

http://web.mit.edu/18.06/www

I hope you will find that page helpful. It is coming close to 30,000 visitors. Please use it freely and suggest how it can be extended and improved.

Chapter 7 connects linear transformations with matrices. The matrix depends on the choice of basis! We show how vectors and matrices change when the basis changes. And we show the linear transformation behind the matrix. I don't start the course with that deeper idea, it is better to understand subspaces first.

Chapter 8 gives important applications—often I choose Markov matrices for a lecture without an exam. Chapter 9 comes back to numerical linear algebra, to explain how $Ax = b$ and $Ax = \lambda x$ are actually solved. Chapter 10 moves from real to complex numbers, as entries in the vectors and matrices. The complete book is appropriate for a two-semester course—it starts gradually and keeps going forward.

Computing in Linear Algebra

The text gives first place to MATLAB, a beautiful system that was developed specifically for linear algebra. This is the primary language of our *Teaching Codes*, written by Cleve Moler for the first edition and extended by Steven Lee for this edition. The Teaching Codes are on the web page, with MATLAB homeworks and references and a short primer. The best way to get started is to solve problems!

We also provide a similar library of Teaching Codes for Maple and Mathematica. The codes are listed at the end of the book, and they execute the same steps that we teach. Then the reader can see matrix theory both ways—the algebra and the algorithms. Those work together perfectly. This textbook supports a course that includes computing and also a course that doesn't.

There is so much good mathematics to learn and to do.

Acknowledgements

This book surely comes with a lot of help. Suggestions arrived by email from an army of readers; may I thank you now! Steven Lee visited M.I.T. three times from Oak Ridge National Laboratory, to teach the linear algebra course 18.06 from this textbook. He created the web page **http://web.mit.edu/18.06/www** and he added new MATLAB Teaching Codes to those written by Cleve Moler for the first edition. (All the Teaching Codes are listed at the end of this book.) I find that these short programs show the essential steps of linear algebra in a very clear way. *Please* look at the web page for help of every kind.

For the creation of the book I express my deepest thanks to five friends. The 1993 first edition was converted to LaTeX2ε by Kai Borre and Frank Jensen in Denmark. Then came the outstanding work by Sueli Rocha on the new edition. First at M.I.T. and then in Hong Kong, Sueli has shared all the excitement and near-heartbreak and eventual triumph that is part of publishing. Vasily Strela succeeded to make the figures print (by somehow reading the postscript file). And in the final crucial step, Amy Hendrickson has done everything to complete the design. She is a professional as well as a friend. I hope you will like the Review of the Key Ideas (at the end of every section) and the clear boxes inside definitions and theorems. The reviews and the boxes highlight the main points, and then my class remembers them.

There is another special part of this book: *The front cover.* A month ago I had a mysterious email message from Ed Curtis at the University of Washington. He insisted that I buy *Great American Quilts: Book 5*, without saying why. Perhaps it is not necessary to admit that I have made very few quilts. On page 131 of that book I found an amazing quilt created by Chris Curtis. She had seen the first edition of this textbook (its cover had slanted houses). They show what linear transformations can do, in Section 7.1. She liked the houses and she made them beautiful. Possibly they became nonlinear, but that's art.

I appreciate that Oxmoor House allowed the quilt to appear on this book. The color was the unanimous choice of two people. And I happily thank Tracy Baldwin for designing her third neat cover for Wellesley-Cambridge Press.

May I now dedicate this book to my grandchildren. It is a pleasure to name the ones I know so far: Roger, Sophie, Kathryn, Alexander, Scott, Jack, William, and Caroline. I hope that all of you will take this linear algebra course one day. *Please pass it, whatever you do*. The author is proud of you.

TABLE OF CONTENTS

1

INTRODUCTION TO VECTORS

The heart of linear algebra is in two operations—both with vectors. We add vectors to get $v + w$. We multiply by numbers c and d to get cv and dw. Combining those operations gives the *linear combination* $cv + dw$.

Chapter 1 explains these central ideas, on which everything builds. We start with two-dimensional vectors and three-dimensional vectors, which are reasonable to draw. Then we move into higher dimensions. The really impressive feature of linear algebra is how smoothly it takes that step into n-dimensional space. Your mental picture stays completely correct, even if drawing a ten-dimensional vector is impossible.

This is where the book is going (into n-dimensional space), and the first steps are the two operations in Sections 1.1 and 1.2:

1.1 ***Vector addition*** $v + w$ and ***linear combinations*** $cv + dw$.

1.2 The ***dot product*** $v \cdot w$ and the ***length*** $\|v\| = \sqrt{v \cdot v}$.

VECTORS AND LINEAR COMBINATIONS ■ 1.1

"You can't add apples and oranges." That sentence might not be news, but it still contains some truth. In a strange way, it is the reason for vectors! If we keep the number of apples separate from the number of oranges, we have a pair of numbers. That pair is a ***two-dimensional vector*** v:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{aligned} v_1 &= \text{number of apples} \\ v_2 &= \text{number of oranges.} \end{aligned}$$

We wrote v as a ***column vector***. The numbers v_1 and v_2 are its "components." The main point so far is to have a single letter v (in boldface) for this pair of numbers v_1 and v_2 (in lightface).

Even if we don't add v_1 to v_2, we do *add vectors*. The first components of v and w stay separate from the second components:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad v + w = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

You see the reason. The total number of apples is $v_1 + w_1$. The number of oranges is $v_2 + w_2$. Vector addition is basic and important. Subtraction of vectors follows the same idea: *The components of* $\boldsymbol{v} - \boldsymbol{w}$ *are* $v_1 - w_1$ *and* _____.

Vectors can be multiplied by 2 or by -1 or by any number c. There are two ways to double a vector. One way is to add $\boldsymbol{v} + \boldsymbol{v}$. The other way (the usual way) is to multiply each component by 2:

$$2\boldsymbol{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} \quad \text{and} \quad -\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

The components of $c\boldsymbol{v}$ are cv_1 and cv_2. The number c is called a "scalar."

Notice that the sum of $-\boldsymbol{v}$ and $\boldsymbol{v}$ is the zero vector. This is $\mathbf{0}$, which is not the same as the number zero! The vector $\mathbf{0}$ has components 0 and 0. Forgive me for hammering away at the difference between a vector and its components. Linear algebra is built on these operations $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$—***adding vectors and multiplying by scalars***.

There is another way to see a vector, that shows all its components at once. The vector $\boldsymbol{v}$ can be represented by an arrow. When $\boldsymbol{v}$ has two components, the arrow is in two-dimensional space (a plane). If the components are v_1 and v_2, the arrow goes v_1 units to the right and v_2 units up. This vector is drawn twice in Figure 1.1. First, it starts at the origin (where the axes meet). This is the usual picture. Unless there is a special reason, our vectors will begin at $(0, 0)$. But the second arrow shows the starting point shifted over to A. The arrows $\overrightarrow{OP}$ and $\overrightarrow{AB}$ represent the same vector. One reason for allowing any starting point is to visualize the sum $\boldsymbol{v} + \boldsymbol{w}$:

Vector addition (head to tail) ***At the end of*** $\boldsymbol{v}$, ***place the start of*** $\boldsymbol{w}$.

We travel along $\boldsymbol{v}$ and then along $\boldsymbol{w}$. Or we take the shortcut along $\boldsymbol{v} + \boldsymbol{w}$. We could also go along $\boldsymbol{w}$ and then $\boldsymbol{v}$. In other words, $\boldsymbol{w} + \boldsymbol{v}$ gives the same answer as $\boldsymbol{v} + \boldsymbol{w}$. These are different ways along the parallelogram (in this example it is a rectangle). The endpoint in Figure 1.2 is the diagonal $\boldsymbol{v} + \boldsymbol{w}$ which is also $\boldsymbol{w} + \boldsymbol{v}$.

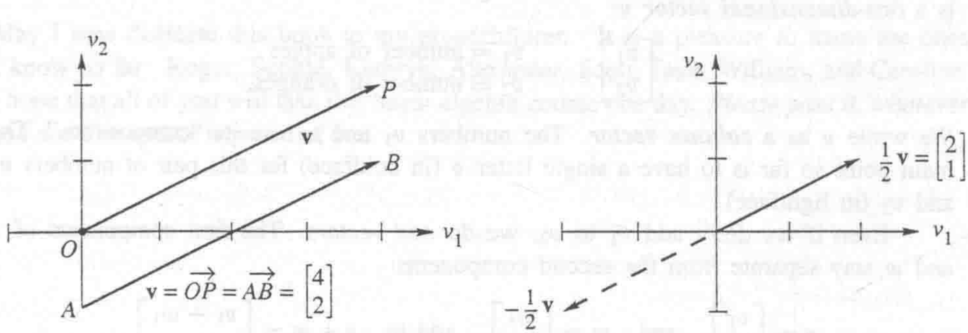


Figure 1.1 The arrow usually starts at the origin $(0, 0)$; $c\boldsymbol{v}$ is always parallel to $\boldsymbol{v}$.

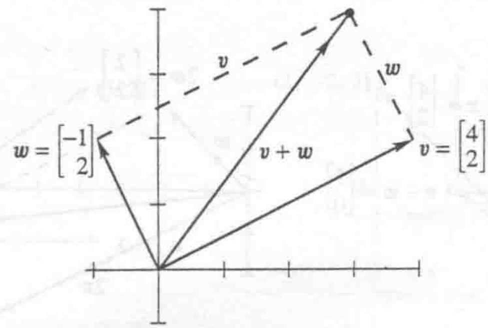


Figure 1.2 Vector addition $v+w$ produces the diagonal of a parallelogram. Add the first components and second components separately.

Check that by algebra: The first component is $v_1 + w_1$ which equals $w_1 + v_1$. The order of addition makes no difference:

$$v + w = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = w + v.$$

The zero vector has $v_1 = 0$ and $v_2 = 0$. It is too short to draw a decent arrow, but you know that $v + 0 = v$. For $2v$ we double the length of the arrow. We reverse its direction for $-v$. This reversing gives a geometric way to subtract vectors.

Vector subtraction To draw $v - w$, go forward along v and then backward along w (Figure 1.3). The components are $v_1 - w_1$ and $v_2 - w_2$.

We will soon meet a "dot product" of vectors. It is not the vector whose components are v_1w_1 and v_2w_2.

Linear Combinations

We have added vectors, subtracted vectors, and multiplied by scalars. The answers $v + w$, $v - w$, and cv are computed a component at a time. By combining these operations, we now form *"linear combinations"* of v and w. Apples still stay separate from oranges—the linear combination in Figure 1.3 is a new vector $cv + dw$.

DEFINITION ***The sum of cv and dw is a linear combination of v and w.***

$$3v + 2w = 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}.$$

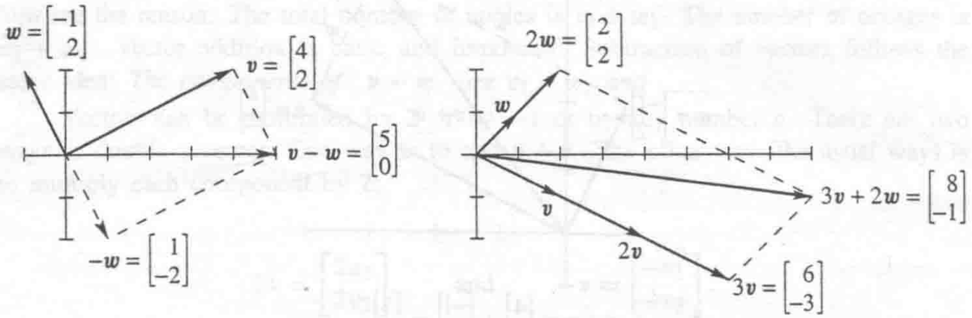


Figure 1.3 Vector subtraction $v - w$ (left). The linear combination $3v + 2w$ (right).

This is the fundamental construction of linear algebra: *multiply and add*. The sum $v + w$ is a special combination, when $c = d = 1$. The multiple $2v$ is the particular case with $c = 2$ and $d = 0$. Soon you will be looking at all linear combinations of v and w—a whole family of vectors at once. It is this big view, going from two vectors to a "plane of vectors," that makes the subject work.

In the forward direction, a combination of v and w is supremely easy. We are given the multipliers $c = 3$ and $d = 2$, so we multiply. Then add $3v + 2w$. The serious problem is the opposite question, when c and d are "unknowns." In that case we are only given the answer: $cv + dw$ has components 8 and -1. We look for the right multipliers c and d. The two components give two equations in these two unknowns.

When 100 unknowns multiply 100 vectors each with 100 components, the best way to find those unknowns is explained in Chapter 2.

Vectors in Three Dimensions

Each vector v with two components corresponds to a point in the xy plane. The components of v are the coordinates of the point: $x = v_1$ and $y = v_2$. The arrow ends at this point (v_1, v_2), when it starts from $(0, 0)$. Now we allow vectors to have three components. The xy plane is replaced by three-dimensional space.

Here are typical vectors (still column vectors but with three components):

$$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad v + w = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

The vector v corresponds to an arrow in 3-space. Usually the arrow starts at the origin, where the xyz axes meet and the coordinates are $(0, 0, 0)$. The arrow ends at the point

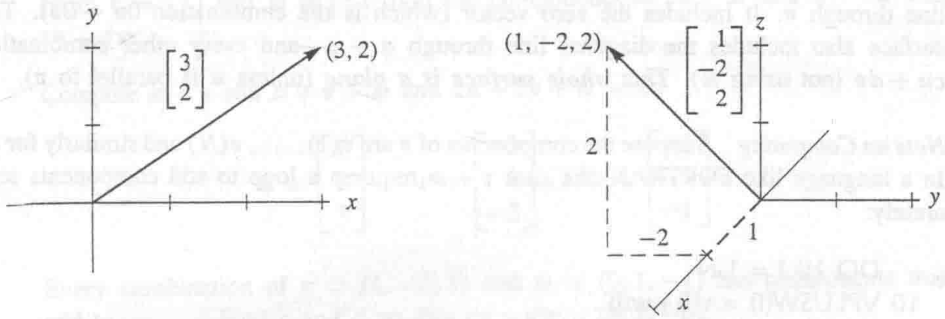


Figure 1.4 Vectors $\begin{bmatrix}x\\y\end{bmatrix}$ and $\begin{bmatrix}x\\y\\z\end{bmatrix}$ correspond to points (x, y) and (x, y, z).

with coordinates $x = 1$, $y = 2$, $z = 2$. There is a perfect match between the ***column vector*** and the ***arrow from the origin*** and the ***point where the arrow ends***. Those are three ways to describe the same vector:

$$\textbf{\textit{From now on}} \quad v = \begin{bmatrix}1\\2\\2\end{bmatrix} \quad \textbf{\textit{is also written as}} \quad v = (1, 2, 2).$$

The reason for the column form (in brackets) is to fit next to a matrix. The reason for the row form (in parentheses) is to save space. This becomes essential for long vectors. To print (1, 2, 2, 4, 4, 6) in a column would waste the environment.

Important note $v = (1, 2, 2)$ is not a row vector. The row vector $[\,1 \;\; 2 \;\; 2\,]$ is absolutely different, even though it has the same three components. It is the "transpose" of v.

A column vector can be printed horizontally (with commas and parentheses). Thus (1, 2, 2) is in actuality a column vector. It is just temporarily lying down.

In three dimensions, vector addition is still done a component at a time. The result $v + w$ has components $v_1 + w_1$ and $v_2 + w_2$ and $v_3 + w_3$—maybe apples, oranges, and pears. You see already how to add vectors in 4 or 5 or n dimensions. This is now the end of linear algebra for groceries!

The addition $v + w$ is represented by arrows in space. When w starts at the end of v, the third side is $v + w$. When w follows v, we get the other sides of a parallelogram. Question: Do the four sides all lie in the same plane? *Yes*. And the sum $v + w - v - w$ goes around the parallelogram to produce ____ .

A typical linear combination of three vectors in three dimensions is $u + 4v - 2w$:

$$\textbf{\textit{Linear combination}} \qquad \begin{bmatrix}1\\0\\3\end{bmatrix} + 4\begin{bmatrix}1\\2\\1\end{bmatrix} - 2\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\9\end{bmatrix}.$$

We end with this question: What surface in 3-dimensional space do you get from all the linear combinations of u and v? The surface includes the line through u and the line through v. It includes the zero vector (which is the combination $0u + 0v$). The surface also includes the diagonal line through $u + v$—and every other combination $cu + dv$ (not using w). ***This whole surface is a plane*** (unless u is parallel to v).

Note on Computing Suppose the components of v are $v(1), \ldots, v(N)$ and similarly for w. In a language like FORTRAN, the sum $v + w$ requires a loop to add components separately:

```
      DO 10 I = 1,N
   10 VPLUSW(I) = v(I)+w(I)
```

MATLAB works directly with vectors and matrices. When v and w have been defined, $v + w$ is immediately understood. It is *printed* unless the line ends in a semicolon. Input two specific vectors as rows—the prime $'$ at the end changes them to columns. Then print $v + w$ and another linear combination:

$$v = [2 \quad 3 \quad 4]' \ ; \quad w = [1 \quad 1 \quad 1]' \ ; \quad u = v + w$$

$$2 * v - 3 * w$$

The sum will print with $u =$. The unnamed combination prints with **ans** =:

$u =$	**ans** =
3	1
4	3
5	5

▪ REVIEW OF THE KEY IDEAS ▪

1. A vector v in two-dimensional space has two components v_1 and v_2.

2. Vectors are added and subtracted a component at a time.

3. The scalar product is $cv = (cv_1, cv_2)$. A linear combination of v and w is $cv + dw$.

4. The linear combinations of two non-parallel vectors v and w fill a plane.

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

1 Draw the vectors $v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $v + w$ and $v - w$ in a single xy plane.

2 If $v + w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $v - w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, compute and draw v and w.

3 From $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the components of $3v + w$ and $v - 3w$ and $cv + dw$.

4 Compute $u + v$ and $u + v + w$ and $2u + 2v + w$ when

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}.$$

5 Every combination of $v = (1, -2, 1)$ and $w = (0, 1, -1)$ has components that add to ____. Find c and d so that $cv + dw = (4, 2, -6)$.

6 In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix} 3 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

7 (a) The subtraction $v - w$ goes forward along v and backward on w. Figure 1.3 also shows a second route to $v - w$. What is it?

(b) If you look at all combinations of v and w, what "surface of vectors" do you see?

8 The parallelogram in Figure 1.2 has diagonal $v + w$. What is its other diagonal? What is the sum of the two diagonals? Draw that vector sum.

9 If three corners of a parallelogram are (1, 1), (4, 2), and (1, 3), what are all the possible fourth corners? Draw two of them.

Problems 10–13 involve the length of vectors. Compute (length of v)^2 as $v_1^2 + v_2^2$.

10 The parallelogram with sides $v = (4, 2)$ and $w = (-1, 2)$ is a rectangle (Figure 1.2). Check the Pythagoras formula $a^2 + b^2 = c^2$ which is for ***right triangles only***:

$$(\text{length of } v)^2 + (\text{length of } w)^2 = (\text{length of } v + w)^2.$$

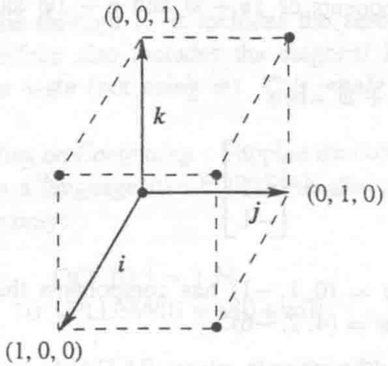
11 In this 90° case, $a^2 + b^2 = c^2$ also works for $v - w$. In Figure 1.2, check that

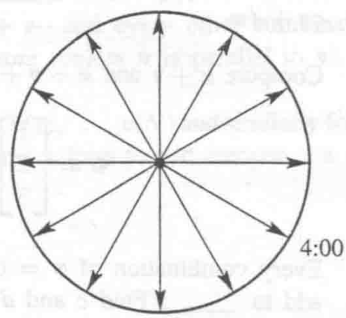
$$(\text{length of } v)^2 + (\text{length of } w)^2 = (\text{length of } v - w)^2.$$

Give an example of v and w (not at right angles) for which this formula fails.

12 To emphasize that right triangles are special, construct v and w without a 90° angle. Compare (length of v)^2 + (length of w)^2 with (length of $v + w$)^2.

13 In Figure 1.2 check that (length of v) + (length of w) is larger than (length of $v + w$). This "triangle inequality" is true for every triangle, except the absolutely thin triangle when v and w are ____. Notice that these lengths are not squared.





Problems 14–18 are about special vectors on cubes and clocks.

14 Copy the cube and draw the vector sum of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$. The addition $\boldsymbol{i} + \boldsymbol{j}$ yields the diagonal of ____.

15 Three edges of the unit cube are $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. Three corners are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$. What are the other five corners and the coordinates of the center point? The center points of the six faces are ____.

16 How many corners does a cube have in 4 dimensions? How many faces? How many edges? A typical corner is $(0, 0, 1, 0)$.

17 (a) What is the sum V of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the vector to 4:00 is removed, find the sum of the eleven remaining vectors.

(c) Suppose the 1:00 vector is cut in half. Add it to the other eleven vectors.

18 Suppose the twelve vectors start from $(0, -1)$ at the bottom of the clock instead of $(0, 0)$ at the center. The vector to 6:00 is zero and the vector to 12:00 is doubled to $(2\boldsymbol{j})$. Add the new twelve vectors.

Problems 19–22 go further with linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (see Figure).

19 The figure shows $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. Mark the points $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$.

20 Mark the point $-\boldsymbol{v} + 2\boldsymbol{w}$ and one other combination $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$. Draw the line of all combinations that have $c + d = 1$.

21 Locate $\frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{2}{3}\boldsymbol{v} + \frac{2}{3}\boldsymbol{w}$. The combinations $c\boldsymbol{v} + c\boldsymbol{w}$ fill out what line? Restricted by $c \geq 0$ those combinations with $c = d$ fill what ray?

22 (a) Mark $\frac{1}{2}\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. Restricted by $0 \leq c \leq 1$ and $0 \leq d \leq 1$, shade in all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.