# C*-ALGEBRA EXTENSIONS AND K-HOMOLOGY

BY

RONALD G. DOUGLAS

ANNALS OF MATHEMATICS STUDIES
PRINCETON UNIVERSITY PRESS

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PRINCETON UNIVERSITY PRESS
AND
UNIVERSITY OF TOKYO PRESS

PRINCETON, NEW JERSEY
1980

Published in Japan exclusively by
University of Tokyo Press
In other parts of the world by
Princeton University Press

Printed in the United States of America
by Princeton University Press, Princeton, New Jersey

Library of Congress Cataloging in Publication data will
be found on the last printed page of this book

# PREFACE

In this book I have written up the Hermann Weyl Lectures, which I gave at the Institute for Advanced Study during February, 1978. My contribution to the work on which I reported was done in collaboration with L. G. Brown of Purdue University and P. A. Fillmore of Dalhousie University. The basic references are [20], [21], [22]. I will not repeat all the references given there although I will give the more important ones and recent papers will be cited in more detail. As we indicated in [20] and [22], there are a number of people to whom we are indebted. I cannot mention them all but would like to acknowledge the influence of M. F. Atiyah and I. M. Singer. In addition, I would like to thank Jerry Kaminker and Claude Schochet for discussions on this material, especially in connection with Chapters five and six. Finally, I would like to express my appreciation to my audience for their interest which spurred me to make this exposition more comprehensive than I had originally planned.

R. G. DOUGLAS

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$C^*$-Algebra Extensions and K-Homology

# CHAPTER 1
# AN OVERVIEW

Although there are no doubt many possible connections between operator theory and algebraic topology, in this book I concentrate on one interplay between the two subjects. The machinery which establishes this connection solves various problems in operator theory which will presently be described and suggests many others. Moreover, exciting applications in algebraic topology seem within reach. We shall say something about these a little later. In this chapter we want to give an overview of our topic including the origins and a general outline of the theory.

Quite appropriately this work can be traced back to a theorem of Hermann Weyl concerning the behavior of the spectrum of a formally self-adjoint differential operator under a change of boundary conditions. A converse due to von Neumann and the evolution of the abstract theories of Fredholm operators and of operator algebras were necessary steps for its development. More recently, the connection between Fredholm operators, index theory, and K-theory developed by Atiyah, Singer, Janich and others set the stage. Finally, the general interest of operator theorists in problems involving compact perturbations provided the particular impetus to this work. These things will be discussed in more detail after the abstract notion which lies at the center of this work is introduced.

We shall study a certain class of $C^*$-extensions

$$0 \to \mathcal{K}(\mathfrak{H}) \to \mathcal{E} \to C(X) \to 0$$

of the $C^*$-algebra $\mathcal{K}(\mathfrak{H})$ of the compact operators by the $C^*$-algebra $C(X)$ of continuous complex-valued functions on a compact metrizable space $X$. Recall that an algebra $\mathcal{E}$ over $\mathbb{C}$ is said to be a *$C^*$-algebra* if $\mathfrak{E}$ possesses a norm $\|\cdot\|$ relative to which $\mathfrak{E}$ is a Banach algebra and an involution $a \to a^*$ satisfying $\|a^*a\| = \|a^*\| \, \|a\|$ for $a$ in $\mathfrak{E}$. If $\mathfrak{H}$ is a complex Hilbert space and $\mathfrak{L}(\mathfrak{H})$ denotes the algebra of bounded linear operators on $\mathfrak{H}$ then $\mathfrak{L}(\mathfrak{H})$ is a $C^*$-algebra with the operator norm and adjoint as involution. More generally, closed, self-adjoint subalgebras of $\mathfrak{L}(\mathfrak{H})$ are $C^*$-algebras and a theorem of Gelfand and Naimark asserts that all $C^*$-algebras up to $^*$-isometrical isomorphism are obtained in this manner. In particular, the algebra $\mathcal{K}(\mathfrak{H})$ of compact operators on $\mathfrak{H}$ is probably the most elementary infinite dimensional $C^*$-algebra. (Recall that $T$ in $\mathfrak{L}(\mathfrak{H})$ is *compact* if the image of the unit ball under $T$ is compact or, equivalently for operators on a Hilbert space, if $T$ is the norm limit of finite rank operators.) Another simple class of $C^*$-algebras consists of those which are commutative and another theorem of Gelfand and Naimark states that these are all $^*$-isometrically isomorphic to $C(X)$ for some compact Hausdorff space $X$. Thus in considering the class of extensions described above we are following a well-established algebraic dictum; that is, study algebras which are obtained as extensions of simpler classes of algebras. However, this was not our original motivation. To explain that let us consider some examples of naturally occurring extensions.

Let $\mathbf{T}$ denote the unit circle in $\mathbf{C}$, $L^2(\mathbf{T})$ the Lebesgue space relative to normalized Lebesgue measure, and $H^2(\mathbf{T})$ the Hardy space obtained as the closure of the analytic polynomials in $L^2(\mathbf{T})$ or equivalently, the closed linear span of $\{z^n : n \geq 0\}$. If $P$ denotes the orthogonal projection from $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$, then for $\psi$ in $C(\mathbf{T})$ the Toeplitz operator $T_\psi$ on $H^2(\mathbf{T})$ is defined by $T_\psi f = P(\psi f)$ for $f$ in $H^2(\mathbf{T})$. If $\mathcal{T}$ denotes the $C^*$-subalgebra of $\mathfrak{L}(H^2(\mathbf{T}))$ generated by $\{T_\psi : \psi \in C(\mathbf{T})\}$ then Coburn showed in [28] that

$$\mathcal{T} = \{T_\psi + K : \psi \in C(\mathbf{T}), K \in \mathcal{K}(H^2(\mathbf{T}))\}$$

and observed that $\mathcal{T}$ is an extension of $\mathcal{K}$ by $C(\mathbf{T})$. That is, there is a short exact sequence

$$0 \longrightarrow \mathcal{K}(H^2(T)) \xrightarrow{i} \mathcal{T} \xrightarrow{\phi} C(T) \longrightarrow 0,$$

where $i$ is inclusion and $\phi$ is the symbol map defined by $\phi(T_\psi + K) = \psi$. It can be shown (cf. [32]) that the basic index results for Toeplitz operators with continuous symbol follow directly from this. Coburn further pointed out that if $\mathcal{S}$ is defined by

$$\mathcal{S} = \{M_\psi + K : \psi \in C(T), K \in \mathcal{K}(L^2(T))\},$$

where $M_\psi$ is the operator defined to be multiplication by $\psi$ on $L^2(T)$, then an extension of $\mathcal{K}$ by $C(T)$ is obtained which is not equivalent to the Toeplitz extension. Thus he raised the problem of determining all extensions of $\mathcal{K}$ by $C(T)$. About the same time Atiyah and Singer arrived at the same problem for reasons which we will explain presently. Before proceeding we enlarge the class of Toeplitz examples.

If $\Omega$ is a strongly pseudo-convex domain in $C^n$, $L^2(\partial\Omega)$ is the Lebesgue space relative to surface measure on $\partial\Omega$, and $H^2(\partial\Omega)$ the Hardy space obtained as the closure in $L^2(\partial\Omega)$ of the functions holomorphic on a neighborhood of the closure of $\Omega$, then the analogue of Toeplitz operators can be defined on $H^2(\partial\Omega)$ (cf. [97], [13]) as the compression $T_\psi$ of a continuous multiplier $\psi$ to $H^2(\partial\Omega)$. (It is not necessary in what follows to know the definition of strongly pseudo-convex but the ball is strongly pseudo-convex while the bidisk is not.) Moreover the sequence

$$0 \longrightarrow \mathcal{K}(H^2(\partial\Omega)) \xrightarrow{i} \mathcal{T}_\Omega \xrightarrow{\phi_\Omega} C(\partial\Omega) \longrightarrow 0$$

can be shown to be exact, where $\mathcal{T}_\Omega$ is the $C^*$-subalgebra of $\mathcal{L}(H^2(\partial\Omega))$ generated by the Toeplitz operators with continuous symbol, $i$ is inclusion, and $\phi_\Omega$ is the symbol map defined by $\phi_\Omega(T_\psi + K) = \psi$. Thus there is a naturally occurring class of Toeplitz extensions. (Before continuing we point out that a similar construction can be carried out using the Lebesgue space relative to volume measure on $\Omega$. The symbol space is still $C(\partial\Omega)$, however.)

In a different direction let M be a closed differentiable manifold and let $L^2(M)$ denote the Lebesgue space defined relative to a fixed smooth measure on M. Now a zero'th order pseudo-differential operator with scalar coefficients defines a bounded operator on $L^2(M)$ and we let $\mathcal{P}_M$ denote the C*-subalgebra of $\mathfrak{L}(L^2(M))$ generated by all such operators together with $\mathcal{K}(L^2(M))$. (I believe adding $\mathcal{K}(L^2(M))$ as generators is unnecessary if M is connected.) Then we obtain the extension

$$0 \longrightarrow \mathcal{K}(L^2(M)) \xrightarrow{i} \mathcal{P}_M \xrightarrow{\phi_M} C(S^*(M)) \longrightarrow 0$$

where i is inclusion, $S^*(M)$ is the cosphere bundle on M, and $\phi_M$ is the symbol map. Moreover, this extension is intimately related to the Atiyah-Singer index theorem and other results, and it was in this connection that Atiyah and Singer were interested in classifying the extensions of $\mathcal{K}$ by C(T).

One of our main sources of motivation sprang from operator theory. Let T be an *essentially normal operator* on $\mathfrak{H}$, that is, an operator T in $\mathfrak{L}(\mathfrak{H})$ such that the self-commutator $[T, T^*] = TT^* - T^*T$ is compact. The problem in operator theory was basically to classify the essentially normal operators up to unitary equivalence modulo $\mathcal{K}$ and to determine the possibilities. Why should one think that such a thing might be possible? To answer that we have to look at the theorem of Weyl to which we alluded earlier. For H a self-adjoint operator, Weyl defined the essential spectrum $\sigma_e(H)$ to be all $\lambda$ in the spectrum $\sigma(H)$ except for isolated eigenvalues of finite multiplicity. He proved in [101] that if the self-adjoint operators $H_1$ and $H_2$ differ by a compact operator, then $\sigma_e(H_1) = \sigma_e(H_2)$. Trivially this implies that if $H_1$ and $H_2$ are self-adjoint operators for which there exists a unitary operator U such that

$$U^*H_1U = H_2 + K$$

for some K in $\mathcal{K}(\mathfrak{H})$, then $\sigma_e(H_1) = \sigma_e(H_2)$. About twenty years later von Neumann established [62] the converse, and almost fifty years later in response to a question of Halmos [41] the result was extended to normal operators by Berg [12]. Thus we have

THEOREM 1. *If* $N_1$ *and* $N_2$ *are normal operators on* $\mathfrak{H}$, *then the following are equivalent: a) there exists a unitary operator* $U$ *and a compact operator* $K$ *such that* $U^*N_1U = N_2 + K$ *and b)* $\sigma_e(N_1) = \sigma_e(N_2)$.

*Moreover, if* $X$ *is any compact subset of* $\mathbb{C}$, *then there exists a normal operator* $N$ *such that* $\sigma_e(N) = X$.

This solved completely the problem of classifying normal operators up to unitary equivalence modulo $\mathcal{K}$ and showed that the equivalence classes are in one to one correspondence with compact subsets of $\mathbb{C}$. (This should be compared with the solution to the unitary equivalence problem for normal operators which involves multiplicity theory and hence equivalence classes correspond to cardinal numbers assigned to a Borel partition of the spectrum.) Therefore it seemed reasonable to try to extend this result to essentially normal operators.

Now for an essentially normal operator $T = N + K$ in the algebraic linear span $\mathcal{N} + \mathcal{K}$, where $\mathcal{N}$ is the collection of normal operators, the only problem is to obtain $\sigma_e(N)$ in terms of $T$. That is easy once it is observed that the essential spectrum of a normal operator coincides with the spectrum in the quotient algebra $\mathfrak{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$. More precisely, since $\mathcal{K}(\mathfrak{H})$ is a closed two-sided ideal in $\mathfrak{L}(\mathfrak{H})$ we can define the quotient $\mathfrak{Q}(\mathfrak{H}) = \mathfrak{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$, usually called the *Calkin algebra*, which can be shown to be a $C^*$-algebra. If $\pi : \mathfrak{L}(\mathfrak{H}) \to \mathfrak{Q}(\mathfrak{H})$ denotes the quotient map, then we can consider the spectrum $\sigma_{\mathfrak{Q}(\mathfrak{H})}(\pi(T))$ for $T$ in $\mathfrak{L}(\mathfrak{H})$ and this spectrum can be shown to coincide with $\sigma_e(T)$ for $T$ normal. Thus the *essential spectrum* for general $T$ is defined to be $\sigma_e(T) = \sigma_{\mathfrak{Q}(\mathfrak{H})}(\pi(T))$. Therefore the Weyl-von Neumann-Berg theorem extends to essentially normal operators $T$ of the form $T = N + K$ since $\sigma_e(N) = \sigma_e(T)$.

Now not all essentially normal operators have this form. The operator $T_z$ on $H^2(\mathbb{T})$ provides such an example but the proof involves the notion of index. Recall that the class of *Fredholm operators* $\mathrm{Fred}(\mathfrak{H})$ can be defined (cf. [32]) such that T is in $\mathrm{Fred}(\mathfrak{H})$ if and only if $\pi(T)$ is invertible in $\mathfrak{Q}(\mathfrak{H})$ if and only 0 is not in $\sigma_e(T)$, or equivalently

$$\mathrm{Fred}(\mathfrak{H}) = \pi^{-1}(\mathfrak{Q}(\mathfrak{H})^{-1}) ,$$

where $\mathfrak{E}^{-1}$ denotes the invertible elements in the algebra $\mathfrak{E}$. Then it follows that

1) $T \in \mathrm{Fred}(\mathfrak{H})$, $K \in \mathfrak{K}(\mathfrak{H})$ implies $T + K \in \mathrm{Fred}(\mathfrak{H})$,
2) $S$, $T \in \mathrm{Fred}(\mathfrak{H})$ implies $ST \in \mathrm{Fred}(\mathfrak{H})$, and
3) $\mathrm{Fred}(\mathfrak{H})$ is an open subset of $\mathfrak{L}(\mathfrak{H})$.

The latter property follows from the fact that the invertible elements in a Banach algebra form an open set. Moreover, a result of Atkinson shows that T is in $\mathrm{Fred}(\mathfrak{H})$ if and only if i) ran T is closed, ii) $\dim \ker T < \infty$, and iii) $\dim \ker T^* < \infty$. Lastly, the *index* is defined from $\mathrm{Fred}(\mathfrak{H})$ to Z by

$$\mathrm{ind}(T) = \dim \ker T - \dim \ker T^*$$

and is a continuous homomorphism which is invariant under compact perturbation.

The relevance of index to our problem lies in the following. By our earlier remarks $[T_z, T_z^*]$ is compact and $\sigma_e(T_z) = \mathbf{T}$; therefore $T_z$ is Fredholm. Further, relative to the orthonormal basis $\{1, z, z^2, \cdots\}$ for $H^2(\mathbf{T})$, $T_z$ is the unilateral shift (that is, $T_z z^n = z^{n+1}$ for $n \geq 0$), while $T_z^*$ is the backward shift. Thus if $(a_0, a_1 \cdots)$ are the Fourier coefficients for f in $H^2(\mathbf{T})$, then $T_z(a_0, a_1 \cdots) = (0, a_0, a_1 \cdots)$ and $T_z^*(a_0, a_1, \cdots) = (a_1, a_2, \cdots)$. Therefore, $\dim \ker T_z = 0$, and $\dim \ker T_z^* = 1$, and hence $\mathrm{ind}\, T_z = -1$. Since $\|Nx\|^2 = (N^*Nx, x) = (NN^*x, x) = \|N^*x\|^2$ for N normal on $\mathfrak{H}$ and x in $\mathfrak{H}$, it follows that $\ker N = \ker N^*$ and hence $\mathrm{ind}\, N = 0$ if N is Fredholm. Therefore, the assumption that $T_z = N + K$ where N is normal and K is compact leads to the contradiction

$$-1 = \mathrm{ind}(T_z) = \mathrm{ind}(N+K) = \mathrm{ind}(N) = 0 .$$

Now not only does index provide us with an example of an essentially normal operator which is not of the form normal plus compact, index is the only other ingredient needed to classify essentially normal operators. This is the main result for essentially normal operators.

THEOREM 2. *Two essentially normal operators* $T_1$ *and* $T_2$ *are unitarily equivalent* modulo $\mathcal{K}$ *if and only if* $\sigma_e(T_1) = \sigma_e(T_2) = X$ and $\operatorname{ind}(T_1-\lambda) = \operatorname{ind}(T_2-\lambda)$ *for* $\lambda$ *in* $\mathbb{C}\setminus X$.

Now what does this have to do with extensions? If $\mathcal{E}_T$ denotes the $C^*$-subalgebra of $\mathcal{L}(\mathfrak{H})$ generated by $I$, $T$ and $\mathcal{K}(\mathfrak{H})$, then the quotient $\mathcal{E}_T/\mathcal{K}(\mathfrak{H})$ is the $C^*$-subalgebra of $\mathcal{Q}(\mathfrak{H})$ generated by $1$ and $\pi(T)$. Since $\mathcal{E}_T/\mathcal{K}(\mathfrak{H})$ is commutative we have

$$\mathcal{E}_T/\mathcal{K}(\mathfrak{H}) \cong C(\sigma_{\mathcal{Q}(\mathfrak{H})}(\pi(T))) = C(\sigma_e(T))$$

by the spectral theorem and we have an extension

$$0 \longrightarrow \mathcal{K}(\mathfrak{H}) \longrightarrow \mathcal{E}_T \xrightarrow{\phi_T} C(\sigma_e(T)) \longrightarrow 0\ ,$$

where $\phi_T$ is the restriction of $\pi$ to $\mathcal{E}_T$. Moreover, one can show that the two extensions $(\mathcal{E}_{T_1}, \phi_{T_1})$ and $(\mathcal{E}_{T_2}, \phi_{T_2})$ are equivalent (in a sense which we make clear in the next chapter) if and only if $T_1$ and $T_2$ are unitarily equivalent modulo $\mathcal{K}$. Furthermore, if $\mathcal{K}(\mathfrak{H}) \subset \mathcal{E} \subset \mathcal{L}(\mathfrak{H})$ and $X \subset \mathbb{C}$ are such that

$$0 \longrightarrow \mathcal{K}(\mathfrak{H}) \xrightarrow{i} \mathcal{E} \xrightarrow{\phi} C(X) \longrightarrow 0$$

is exact, then any $T$ in $\mathcal{E}$ is essentially normal. Moreover, if $\phi(T) = z$, then $\sigma_e(T) = X$ and $(\mathcal{E}_T, \phi_T)$ is equivalent to $(\mathcal{E}, \phi)$. Thus all extensions of $\mathcal{K}$ by $C(X)$ arise from essentially normal operators for $X \subset \mathbb{C}$. Therefore the problem of classifying essentially normal operators is equivalent to that of classifying extensions.

Thus there are various reasons to be interested in extensions of $\mathcal{K}$ by $C(X)$. In our study of these extensions the equivalence classes for fixed $X$ are shown to form an abelian group $\operatorname{Ext}(X)$ such that $X \mapsto \operatorname{Ext}(X)$ defines a homotopy invariant covariant functor. Moreover, this functor can be used to define a generalized Steenrod homology theory which is dual to K-theory. Let me conclude this chapter by showing how this proves the theorem stated earlier.

One of the pairings we define between Ext and K-theory yields a homomorphism

$$\gamma_\infty : \operatorname{Ext}(X) \to \operatorname{Hom}(K^1(X), \mathbb{Z})$$

which we show is an isomorphism for $X \subset \mathbb{C}$. In this special case we will write $\gamma_1$ for this homomorphism. Since $X \subset \mathbb{C}$ we have

$$K^1(X) = H^1(X, \mathbb{Z}) = \pi^1(X) ,$$

where $\pi^1(X)$ is the first cohomotopy group of $X$ or the group relative to pointwise multiplication of homotopy classes of maps from $X$ into the nonzero complex numbers $\mathbb{C}^*$. The definition of

$$\gamma_1 : \operatorname{Ext}(X) \to \operatorname{Hom}(\pi^1(X), \mathbb{Z})$$

goes as follows: fix $(\mathcal{E}, \phi)$ in $\operatorname{Ext}(X)$ and for $f : X \to \mathbb{C}^*$ we define

$$\gamma_1(\mathcal{E}, \phi)[f] = \operatorname{ind}(\phi^{-1}(f)) ,$$

where $[f]$ denotes the element of $\pi^1(X)$ defined by $f$. To see that this is well defined, note that $\phi^{-1}(f)$ is not a unique Fredholm operator but has a well-defined index which depends only on the homotopy class of $f$. It is easy to check that $\gamma_1$ is a homomorphism. Now for $X \subset \mathbb{C}$ what is $\pi^1(X)$? If we let $\mathbb{C} \setminus X = O_\infty \cup O_1 \cup O_2 \cup \cdots$ denote the components, where $O_\infty$ is the unbounded one, then $\pi^1(X)$ is the free abelian group with one generator for each bounded component. For the extension $(\mathcal{E}_T, \phi_T)$, the homomorphism $\gamma_1(\mathcal{E}_T, \phi_T)$ is defined by $[O_i] \to n_i$, where $n_i = \operatorname{ind}(T - \lambda_i)$ for some $\lambda_i$ in $O_i$, and now Theorem 2 is obvious. Moreover, since $\gamma_1$ is surjective, the equivalence classes of essentially normal operators with essential spectrum X are obtained by prescribing arbitrary integers for the bounded components of the complement of X in C.

Equivalently, since

$$\mathrm{Hom}(\pi^1(X), Z) \cong \mathrm{Hom}(H^1(X, Z), Z) \cong H^0(C \setminus X, Z) \cong [C \setminus X, Z]$$

by Steenrod duality (where $[C \setminus X, Z]$ denotes the group of locally constant integer-valued functions defined on $C \setminus X$) $\gamma_1$ can be defined by

$$\gamma_1(\mathcal{E}_T, \phi_T)(\lambda) = \mathrm{ind}(T-\lambda I)$$

and Theorem 2 is obvious.

Although our proof certainly involves operator theory, it also involves a critical use of ideas and techniques from algebraic topology and homological algebra. Moreover, there is presently no proof of these results which does not.[1] In fact there is no purely operator theoretic proof of either of the following corollaries

COROLLARY. *An essentially normal operator* T *is in* $\mathfrak{N} + \mathcal{K}$ *if and only if* $\mathrm{ind}(T-\lambda) = 0$ *for* $\lambda$ *in* $C \setminus \sigma_e(T)$.

COROLLARY. *The collection* $\mathfrak{N} + \mathcal{K}$ *is norm-closed.*

The situation in several variables is more subtle. For example, the analogue of the last corollary is false for compact perturbations of commuting pairs of normal operators. More precisely, the collection

$$\{N_1+K_1, N_2+K_2 : N_1, N_2 \in \mathfrak{N}, [N_1, N_2] = 0, K_1, K_2 \in \mathcal{K}\}$$

---

[1] In [30] Davie gives an exposition of the proofs of these results in which the ideas from operator theory and algebraic topology are separated as much as possible and the latter kept to a minimum.

is not norm closed. Why is that the case? Certain types of topological pathologies cannot occur in $\mathbf{C}$ but do in $\mathbf{C}^2$ (and all $\mathbf{C}^n$ for $n \geq 2$). Here we are using the fact that the generalized homology theory defined by Ext is not continuous. Thus Ext is sensitive enough to detect these pathologies.

The existence of the above pathology for pairs causes me to doubt that a purely operator theoretic proof of the last corollary is possible.