



国外优秀数学著作  
原 版 系 列

解析数论问题集  
Problems in Analytic Number Theory

(第二版)

[美] 默尔蒂 (Murty, M. R.) 著

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U. S. R. Murty

Problems in Analytic Number Theory

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# Preface to the Second Edition

This expanded and corrected second edition has a new chapter on the important topic of equidistribution. Undoubtedly, one cannot give an exhaustive treatment of the subject in a short chapter. However, we hope that the problems presented here are enticing that the student will pursue further and learn from other sources.

A problem style presentation of the fundamental topics of analytic number theory has its virtues, as I have heard from those who benefited from the first edition. Mere theoretical knowledge in any field is insufficient for a full appreciation of the subject and one often needs to grapple with concrete questions in which these ideas are used in a vital way. Knowledge and the various layers of “knowing” are difficult to define or describe. However, one learns much and gains insight only through practice. Making mistakes is an integral part of learning. Indeed, “it is practice first and knowledge afterwards.”

Kingston, Ontario, Canada, September 2007

M. Ram Murty



# Acknowledgments for the Second Edition

I would like to thank several people who have assisted me in correcting and expanding the first edition. They are Amir Akbary, Robin Chapman, Keith Conrad, Chantal David, Brandon Fodden, Sanoli Gun, Wentang Kuo, Yu-Ru Liu, Kumar Murty, Purusottam Rath and Michael Rubinstein.

Kingston, Ontario, Canada, September 2007

M. Ram Murty



# Preface to the First Edition

"In order to become proficient in mathematics, or in any subject," writes André Weil, "the student must realize that most topics involve only a small number of basic ideas." After learning these basic concepts and theorems, the student should "drill in routine exercises, by which the necessary reflexes in handling such concepts may be acquired. . . . There can be no real understanding of the basic concepts of a mathematical theory without an ability to use them intelligently and apply them to specific problems." Weil's insightful observation becomes especially important at the graduate and research level. It is the viewpoint of this book. Our goal is to acquaint the student with the methods of analytic number theory as rapidly as possible through examples and exercises.

Any landmark theorem opens up a method of attacking other problems. Unless the student is able to sift out from the mass of theory the underlying techniques, his or her understanding will only be academic and not that of a participant in research. The prime number theorem has given rise to the rich Tauberian theory and a general method of Dirichlet series with which one can study the asymptotics of sequences. It has also motivated the development of sieve methods. We focus on this theme in the book. We also touch upon the emerging Selberg theory (in Chapter 8) and  $p$ -adic analytic number theory (in Chapter 10).



This book is a collection of about five hundred problems in analytic number theory with the singular purpose of training the beginning graduate student in some of its significant techniques. As such, it is expected that the student has had at least a semester course in each of real and complex analysis. The problems have been organized with the purpose of self-instruction. Those who exercise their mental muscles by grappling with these problems on a daily basis will develop not only a knowledge of analytic number theory but also the discipline needed for self-instruction, which is indispensable at the research level.

The book is ideal for a first course in analytic number theory either at the senior undergraduate level or the graduate level. There are several ways to give such a course. An introductory course at the senior undergraduate level can focus on chapters 1, 2, 3, 9, and 10. A beginning graduate course can in addition cover chapters 4, 5, and 8. An intense graduate course can easily cover the entire text in one semester, relegating some of the routine chapters such as chapters 6, 7, and 10 to student presentations. Or one can take up a chapter a week during a semester course with the instructor focusing on the main theorems and illustrating them with a few worked examples.

In the course of training students for graduate research, I found it tedious to keep repeating the cyclic pattern of courses in analytic and algebraic number theory. This book, along with my other book "Problems in Algebraic Number Theory" (written jointly with J. Esmonde), which appears as *Graduate Texts in Mathematics*, Vol. 190, are intended to enable the student gain a quick initiation into the beautiful subject of number theory. No doubt, many important topics have been left out. Nevertheless, the material included here is a "basic tool kit" for the number theorist and some of the harder exercises reveal the subtle "tricks of the trade."

Unless the mind is challenged, it does not perform. The student is therefore advised to work through the questions with some attention to the time factor. "Work expands to fill the time allotted to it" and so if no upper limit is assigned, the mind does not get focused. There is no universal rule on how long one should work on a problem. However, it is a well-known fact that self-discipline, whatever shape it may take, opens the door for inspiration. If the mental muscles are exercised in this fashion, the nuances of the solution

become clearer and significant. In this way, it is hoped that many, who do not have access to an “external teacher” will benefit by the approach of this text and awaken their “internal teacher.”

Princeton, November 1999

M. Ram Murty



# Acknowledgments for the First Edition

I would like to thank Roman Smirnov for his excellent job of typesetting this book into L<sup>A</sup>T<sub>E</sub>X. I also thank Amir Akbary, Kalyan Chakraborty, Alina Cojocaru, Wentang Kuo, Yu-Ru Liu, Kumar Murty, and Yiannis Petridis for their comments on an earlier version of the manuscript. The text matured from courses given at Queen's University, Brown University, and the Mehta Research Institute. I thank the students who participated in these courses. Since it was completed while the author was at the Institute for Advanced Study in the fall of 1999, I thank IAS for providing a congenial atmosphere for the work. I am grateful to the Canada Council for their award of a Killam Research Fellowship, which enabled me to devote time to complete this project.

Princeton, November 1999

M. Ram Murty



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