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Hodge Theory and Complex Algebraic Geometry II

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HODGE THEORY AND COMPLEX ALGEBRAIC GEOMETRY II

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Introduction

The first volume of this book was devoted to the study of the cohomology of compact Kähler manifolds. The main results there can be summarised as follows. (Throughout this volume, we write for example vI.6.1 to refer to volume I, section 6.1.)

The Hodge decomposition (vI.6.1). If X is a compact Kähler manifold, then for each integer k, we have a canonical decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

known as the Hodge decomposition, depending only on the complex structure of X. Every space $H^{p,q}(X) \subset H^k(X, \mathbb{C})$ can be identified with the set of cohomology classes representable in de Rham cohomology by a closed form which is of type (p, q) at every point of X, relative to the complex structure on X. In particular, we have the Hodge symmetry

$$H^{p,q}(X) = \overline{H^{q,p}}(X),$$

where $\alpha \mapsto \overline{\alpha}$ denotes the natural action of complex conjugation on $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$. The Hodge filtration F on $H^k(X, \mathbb{C})$ is the decreasing filtration defined by

$$F^i H^k(X, \mathbb{C}) = \bigoplus_{p \geq i} H^{p,k-p}(X).$$

The Lefschetz decomposition (vI.6.2). Let ω be a Kähler form on X. Then ω is a real closed 2-form of class $[\omega] \in H^2(X, \mathbb{R})$. We write

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$$

for the operator (known as the Lefschetz operator) obtained by taking the cup-product with the class $[\omega]$. For $n = \dim_{\mathbb{C}} X$, and for every $k \leq n$, we have isomorphisms

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

(this result is known as the hard Lefschetz theorem), and thus we have the Lefschetz decomposition

$$H^k(X, \mathbb{R}) = \bigoplus_{2r \leq k} L^r H^{k-2r}(X, \mathbb{R})_{\text{prim}}, \quad k \leq n,$$

where the primitive cohomology $H^l(X, \mathbb{R})_{\text{prim}}$ for $l \leq n$ is defined by

$$H^l(X, \mathbb{R})_{\text{prim}} = \operatorname{Ker}(L^{n-l+1} : H^l(X, \mathbb{R}) \to H^{2n-l+2}(X, \mathbb{R})).$$

Mixed Hodge structures (vI.8.4). Let X be a compact Kähler manifold, and let $Z \subset X$ be a closed analytic subset. Let U be the open set $X - Z$. Then the cohomology groups $H^k(U, \mathbb{Q})$ are equipped with a mixed Hodge structure of weight n, i.e. with two filtrations W and F, an increasing filtration W defined over $\mathbb{Q}$, and a decreasing filtration F defined over $\mathbb{C}$, satisfying the condition:

The filtration F_i induced by F on each space $K_i := \mathrm{Gr}_i^W H^k(U, \mathbb{C})$ equips K_i with a pure Hodge structure of weight $n + i$.

This means that for every integer p, it satisfies the condition

$$F_i^p K_i \oplus \overline{F}_i^{n+i+1-p} K_i = K_i,$$

which implies the existence of a Hodge decomposition

$$K_i = \bigoplus_{p+q=n+i} K_i^{p,q}, \quad K_i^{p,q} = F_i^p K_i \cap \overline{F}_i^{n+i-p} K_i.$$

Variations of Hodge structure (vI.10.1). If $\phi : X \to Y$ is a proper holomorphic submersive map with Kähler fibres, the Hodge filtration on the cohomology of the fibres X_y of ϕ varies holomorphically in the following sense. By Ehresmann's theorem, locally over each point $0 \in Y$, the fibration ϕ admits differentiable trivialisations

$$F = (F_0, \phi) : X_U \cong X_0 \times U, \quad X_U := \phi^{-1}(U).$$

The map F_0 is a retraction of X onto the fibre X_0, and for each $y \in U$, it induces a diffeomorphism $X_y \cong X_0$. In particular, we have a canonical isomorphism when U is contractible, namely the isomorphism

$$H^k(X_y, \mathbb{Z}) \cong H^k(X_0, \mathbb{Z})$$

obtained by combining the two restriction isomorphisms

$$H^k(X_U, \mathbb{Z}) \cong H^k(X_0, \mathbb{Z}) \quad \text{and} \quad H^k(X_U, \mathbb{Z}) \cong H^k(X_y, \mathbb{Z}).$$

Letting r_i denote the integer $\dim F^i H^k(X_y, \mathbb{C})$ for all $y \in Y$, then for each integer i, we have the map

$$\mathcal{P} : U \to \mathrm{Grass}(r_i, H^k(X_0, \mathbb{C})),$$

which to $y \in U$ associates the subspace

$$F^i H^k(X_y, \mathbb{C}) \subset H^k(X_y, \mathbb{C}) = H^k(X_0, \mathbb{C}).$$

The fact that the Hodge filtration varies holomorphically with the complex structure on the fibres can be expressed by the fact that the so-called period map $\mathcal{P}$ is holomorphic for every k, i.

Transversality (vI.10.2). The period map defined above locally gives a holomorphic subbundle

$$F^i \mathcal{H}^k \subset \mathcal{H}^k,$$

where $\mathcal{H}^k = H^k(X_0, \mathbb{C}) \otimes \mathcal{O}_U$ is the sheaf of sections of the trivial holomorphic vector bundle with fibre $H^k(X_0, \mathbb{C})$. Let $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_U$ be the connection given by the usual differentiation of functions in the trivialisation above.

The Griffiths transversality condition is, without a doubt, the most important notion in the theory of variations of Hodge structure. It states that the Hodge bundles $F^i \mathcal{H}^k$ satisfy the property

$$\nabla F^i \mathcal{H}^k \subset F^{i-1} \mathcal{H}^k \otimes \Omega_Y.$$

Note that the data $(\mathcal{H}^k,\ F^i \mathcal{H}^k,\ \nabla)$ are in fact globally defined on Y, but they are only locally trivial; ∇ is known as the Gauss–Manin connection. In general, the Hodge bundle will be defined by

$$\mathcal{H}^k = H^k_{\mathbb{C}} \otimes \mathcal{O}_Y,$$

where $H^k_{\mathbb{C}} = R^k \phi_* \mathbb{C}$. The isomorphisms used above,

$$H^k_{\mathbb{C}}(U) \cong H^k(X_y, \mathbb{C}) \quad \text{for} \quad y \in U,$$

simply show that $H^k_{\mathbb{C}}$ is a local system, and give local trivialisations $H^k_{\mathbb{C}}$ of $\mathcal{H}^k$.

Cycle classes and the Abel–Jacobi map (vI.11.1, vI.12.1). Let $Z \subset X$ be a closed, reduced and irreducible analytic subset of codimension k of a compact Kähler manifold X. We have the cohomology class $[Z] \in H^{2k}(X, \mathbb{Z})$, which can be defined, for example, as the Poincaré dual class of $j_*[\widetilde{Z}]_{\mathrm{fund}}$, where $j : \widetilde{Z} \to Z \to X$ is a desingularisation of Z and $[\widetilde{Z}]_{\mathrm{fund}} \in H_{2\dim \widetilde{Z}}(\widetilde{Z}, \mathbb{Z})$ is the homology class of the smooth compact oriented manifold $\widetilde{Z}$. Then the image of the class $[Z]$ in $H^{2k}(X, \mathbb{C})$ lies in $H^{k,k}(X)$. Such a class is called a Hodge class.

Using Hodge theory, one can also define secondary invariants, called Abel–Jacobi invariants, for a cycle $Z = \sum_i n_i Z_i$ of codimension k which is homologous to 0, i.e. which is such that $\sum_i n_i [Z_i] = 0$ in $H^{2k}(X, \mathbb{Z})$. The Hodge decomposition gives a decomposition

$$H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}.$$

We then define the kth intermediate Jacobian of X as the complex torus

$$J^{2k-1}(X) = H^{2k-1}(X, \mathbb{C})/(F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z})),$$

and we have the Abel–Jacobi invariant

$$\Phi_X^k(Z) \in J^{2k-1}(X)$$

defined by Griffiths. The Abel–Jacobi map generalises the Albanese map for 0-cycles given by

$$\mathrm{alb}_X : \mathcal{Z}_0(X)_{\mathrm{hom}} \to J^{2n-1}(X) = H^0(X, \Omega_X)^*/H_1(X, \mathbb{Z}), \qquad n = \dim X$$

$$z \mapsto \int_\gamma \in H^0(X, \Omega_X)^*, \quad \partial\gamma = z.$$

These results highlight the existence of relations between Hodge theory, topology, and the analytic cycles of a Kähler manifold. For example, the Hodge decomposition and the Hodge symmetry show that the Betti numbers $b_i(X) = \operatorname{rank} H^i(X, \mathbb{Z})$ are even whenever i is odd. The hard Lefschetz theorem shows that the Betti numbers b_{2i} are increasing for $2i \le n = \dim X$, and that the Betti numbers b_{2i-1} are increasing for $2i - 1 \le n = \dim X$. The cycle class map shows that the existence of interesting analytic cycles of codimension k is related to the existence of Hodge classes of degree $2k$, which can be seen on the Hodge structure on $H^{2k}(X)$. Finally, in the algebraic case, where we may assume that the class $[\omega]$ is integral and is even the cohomology class of a hypersurface $Y \subset X$, the hard Lefschetz theorem partly implies the Lefschetz theorem on hyperplane sections, which says that if $j : Y \hookrightarrow X$ is the inclusion of an ample hypersurface, then the restriction map

$$j^* : H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$$

is an isomorphism for $k < \dim Y$ and an injection for $k = \dim Y$. Indeed, by Kodaira's embedding theorem, the ampleness of Y is equivalent to the condition that the real cohomology class $[Y] \in H^2(X, \mathbb{R})$ is a Kähler class. As we have the equalities

$$j_* \circ j^* = L, \quad j^* \circ j_* = L_Y,$$

where L (resp. L_Y) is the Lefschetz operator associated to the Kähler class $[Y]$ (resp. $[Y]_{|Y}$), the hard Lefschetz theorem shows for example that the restriction map $j^* : H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$ is injective for $k \leq \dim Y$ and surjective for $k > \dim Y$.

The fact that the period map is holomorphic also gives relations between Hodge theory and algebraic geometry. For example, it enables us (at least partially) to study moduli spaces classifying the deformations of the complex structure on a polarised algebraic variety, and possibly, when the period map is injective, to realise these moduli spaces as subspaces of domains of global periods. Other subtler applications of the fact that the period map is holomorphic come from the study of the curvature of the Hodge bundles, which can make it possible to polarise the moduli space itself (see Viehweg 1995; Griffiths 1984). Finally, we also deduce that for a family of smooth projective or compact Kähler varieties $\phi : X \to Y$, the Hodge loci $Y_\lambda^k \subset Y$ for a section λ of the local system $R^{2k}\phi_*\mathbb{Z}$, which are defined by

$$Y_\lambda^k = \{y \in Y \mid \lambda_y \in F^k H^{2k}(X_y, \mathbb{C})\},$$

are analytic subsets of Y. This result agrees with the Hodge conjecture, which predicts that $y \in Y_\lambda^k$ if and only if a multiple of λ_y is the cohomology class of a cycle $Z_y \subset X_y$ of codimension k, so that Y_λ^k is the image in Y of a relative Hilbert scheme parametrising subvarieties in the fibres of ϕ.

The applications described above do not constitute particularly tight links between the topology of algebraic varieties, their algebraic cycles and their Hodge theory. The present volume is devoted to the description of much finer interactions between these three domains. We do not, however, propose an exhaustive description of these interactions here, and each of the three parts of this volume ends with a sketch of possible developments which lie beyond the scope of this course. The remainder of this introduction aims to give a synthetic picture of these interactions, which might otherwise be obscured by the separation of the volume into three seemingly independent parts.

Two themes which recur constantly throughout this volume are the Lefschetz theorems and Leray spectral sequences. In the first case, we compare the topology of an algebraic variety X with that of its hyperplane sections, and in the second case we study the topology of a variety X admitting a (usually proper and submersive) morphism $\phi : X \to Y$, using the topology of the fibres X_y, and more precisely in the submersive case, using the local systems $R^k\phi_*\mathbb{Z}$ on Y.

The Lefschetz theorem on hyperplane sections is proved using Morse theory on affine varieties, and does not require any arguments from Hodge theory. However, it does not yield the hard Lefschetz theorem, i.e. the Lefschetz decomposition, which is the only ingredient needed (in an entirely formal way) in the proof of Deligne's theorem:

Theorem 0.1 *The Leray spectral sequence of the rational cohomology of a projective fibration degenerates at E_2.*

Concretely, this result implies the following invariant cycles theorem for smooth projective fibrations:

Theorem 0.2 *If $\phi : X \to Y$ is a smooth projective fibration, then the restriction map*

$$H^k(X, \mathbb{Q}) \to H^k(X_y, \mathbb{Q})^\rho$$

is surjective.

Here, $H^k(X_y, \mathbb{Q})^\rho \subset H^k(X_y, \mathbb{Q})$ denotes the subspace of classes invariant under the monodromy action

$$\rho : \pi_1(Y, y) \to \operatorname{Aut} H^k(X_y, \mathbb{Q}).$$

This puts important constraints on the families of projective varieties. However, qualitatively speaking, it is not a very refined statement. Rather, it is Hodge theory which yields the true global invariant cycles theorem, which imposes qualitative constraints on the monodromy representation associated to a projective fibration. If $\phi : X \to Y$ is a dominant morphism between smooth projective varieties, and $U \subset Y$ is the Zariski open (dense) subset of regular values of ϕ, then we have a smooth and proper fibration $\phi : X_U := \phi^{-1}(U) \to U$, so we have a monodromy representation

$$\rho : \pi_1(U, y) \to \operatorname{Aut} H^k(X_y, \mathbb{Q}) \quad \text{for} \quad y \in U.$$

Then, we have the following result.

Theorem 0.3 *The restriction map*

$$H^k(X, \mathbb{Q}) \to H^k(X_y, \mathbb{Q})^\rho \quad \text{for} \quad y \in U.$$

is surjective. In particular, $H^k(X_y, \mathbb{Q})^\rho$ is a Hodge substructure of $H^k(X_y, \mathbb{Q})$.

The main additional ingredient enabling us to deduce this theorem from the preceding one is the existence of mixed Hodge structures on the cohomology groups of a quasi-projective complex manifold, and the strictness of the morphisms of mixed Hodge structures.

These results, which illustrate the qualitative influence of Hodge theory on the topology of algebraic varieties, are the main object of the Part I of this volume, which is devoted to topology. It also contains an exposition of Picard–Lefschetz theory, which gives a precise description of the geometry of a Lefschetz degeneration. If $Y \overset{j}{\hookrightarrow} X$ is the inclusion of a smooth hyperplane section, the vanishing cohomology $H^*(Y, \mathbb{Z})_{\mathrm{van}}$ is defined as the kernel of the Gysin morphism

$$j_* : H^*(Y, \mathbb{Z}) \to H^{*+2}(X, \mathbb{Z}).$$

Picard–Lefschetz theory shows that the vanishing cohomology is generated by the vanishing cycles, which are classes of spheres contracting to a point when Y degenerates to a nodal hypersurface. Another important consequence of this study is the description of the local monodromy action (the Picard–Lefschetz formula). Combined with the preceding result, it gives the irreducibility theorem for the monodromy action on the vanishing cohomology for the universal family of smooth hyperplane sections of a smooth projective variety X.

This result has numerous consequences, in particular in the study of algebraic cycles; it is a key ingredient in Lefschetz' proof of the Noether–Lefschetz theorem, which says that the Picard group of a general surface Σ of degree ≥ 4 in $\mathbb{P}^3$ is generated by the class of the line bundle $\mathcal{O}_\Sigma(1)$. It also occurs in the proof of the Green–Voisin theorem on the triviality of the Abel–Jacobi map for general hypersurfaces of degree ≥ 6 in $\mathbb{P}^4$. Using the Picard–Lefschetz formula and the transitivity of the monodromy action on vanishing cycles, one can also show that the monodromy group is very large; indeed, it tends to be equal to the group of isomorphisms preserving the intersection form (see Beauville 1986b). This has important restrictive consequences on the Hodge structures of general hyperplane sections: apart from the applications mentioned above, Deligne (1972) uses the monodromy group (combined with the notion of the Mumford group of a Hodge structure) to show that the rational Hodge structure on the H^2 of a general surface of degree ≥ 5 in $\mathbb{P}^3$ is not a quotient of the Hodge structure on the H^2 of an abelian variety. All these results illustrate the influence of topology on Hodge theory.

The second part of this volume is devoted to the study of infinitesimal variations of Hodge structure for a family of smooth projective varieties $\phi : X \to Y$, and its applications, especially those concerning the case of complete families of hypersurfaces or complete intersections of a given variety X.