Studies in the Development of Modern Mathematics
Volume 3

FOURIER SERIES AND WAVELETS

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and
Pierre-Gilles Lemarié-Rieusset

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FOURIER SERIES AND WAVELETS

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Fourier Series and Wavelets
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PREFACE

Like the others in this collection, the present book has different aspects: history, classical mathematics and contemporary mathematics. The period of time covered extends from 1807, when Fourier wrote his first memoir on the Analytic Theory of Heat, to 1994 and the latest developments on wavelets. The work is divided into two parts. The first, written by Jean-Pierre Kahane, deals with Fourier series in the classical sense: decomposition of a function into harmonic components. The second, by Pierre-Gilles Lemarié-Rieusset, expounds the modern theory of wavelets, the most recent tool in pure and applied harmonic analysis. There is an interplay between these two topics. Some common features appear in their history, their linkage with physics and numerical computation, their role and impact in mathematics. As the first part is more classical, emphasis was put on the historical aspect; how problems appear and move in the course of time. The history is shorter in the second part, and thus the purely mathematical exposition – including original contributions – plays a central role.

From the Fourier point of view, mathematical analysis originates from the study of Nature and expresses natural laws in the most general and powerful way. At first, Fourier series are a general method, including a good numerical algorithm, to describe and to compute the functions which occur in the heat diffusion and equilibrium. Then they become an interesting object by themselves and the germ of new theories, developed by the followers of Fourier. In succession we see Dirichlet and the convergence problem, Riemann and real analysis, Cantor and set theory, Lebesgue and functional analysis, probabilistic methods and algebraic structures. Classical Fourier series are still a seminal branch of modern mathematics, as well as a tool used constantly by physicists and engineers. The fast Fourier transform has extended this use enormously in the past thirty years.

Interaction with physics and the construction of efficient algorithms for numerical computation, which appear in Fourier series from the very beginning, are also at the heart of wavelet theory. Here the initiators were engineers and physicists, and mathematicians came later. But in no time wavelets became a unifying language and method both outside and inside mathematics. Now they play a decisive role in the new network which expands between mathematical analysis, theoretical physics, signal analysis, image analysis, telecommunications and fast methods of computation, thanks to which new applications were found for purely mathematical theories.

The book is meant to give an idea of these movements as well as solid information on Fourier series and the state of the art about wavelets. On these matters the authors have personal experience and personal views. This is clear in the choice of the original papers by Fourier, Dirichlet, Riemann and Cantor, which are reproduced in the first part of the book, as well as in the choice and treatment of purely mathematical questions, in both the first and second parts.

The authors are grateful to a number of colleagues and collaborators for their help in scientific, linguistic or bibliographic matters, among others Fan Ai-hua, Olivier Gebuhrer, Monique Hakim, Geoffrey Howson, Lee Lorch, Yves Meyer, Hélène Nocton, Hervé Queffélec, Jean-Bernard Robert, Jan Stegeman and Guido Weiss.

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The figures of Part II have been drawn with help of MICRONDE, a software package developed at Orsay by Y. and M. Misiti, G. Oppenheim and J.M. Poggi as a preliminary version of a MATLAB wavelet toolbox.

CONTENTS

Introduction

WHAT ARE FOURIER SERIES ABOUT ?

The subject matter of Fourier series consists essentially of two formulas :

$$f(x) = \sum c_n e^{inx}, \tag{1}$$

$$c_n = \int f(x) e^{-inx} \frac{dx}{2\pi}. \tag{2}$$

The first involves a series and the second an integral.

We can look at these formulas in two ways. First, we can start with a convergent trigonometric series and define $f(x)$ as its sum. What kind of functions do we obtain through formula (1)? Given such a function, are the coefficients well defined? Can they be computed by formula (2)? This circle of ideas was initiated by Riemann and continued by Cantor, Lebesgue and Denjoy. Strikingly enough, the Riemann integral, the Lebesgue integral and the Denjoy integral were either introduced in relation to or immediately applied to this specific question. It is also the core of the Cantor theory of uniqueness of trigonometric series, the source of the Cantor theory of transfinite induction. In the 20th century this was continued mainly by the Polish and Russian schools. Trigonometric series, as a subject in pure mathematics, provide an interesting and constant interplay among different theories: functions of a real variable, functions of a complex variable, theory of sets and theory of numbers.

Secondly, we can start with a function $f(x)$, apply formula (2) and look at the series in (1). Does it actually converge to $f(x)$? This is the point of view of Fourier. Fourier says that (2) can be applied to an arbitrary function, and that it is possible to prove that the series in (1) converges to that function. He was wrong: first, the integrable functions should be defined; then, the convergence of Fourier series is a delicate matter, where difficult theorems coexist with strange counterexamples. The names of Dirichlet, du Bois-Reymond, Kolmogorov and Carleson, are associated with the main landmarks in this respect. However, the intuition of Fourier was essentially right. Given a periodic phenomenon f, with period 2π (it can be a function, a measure, a distribution in the sense of Schwartz, maybe something else), it is simply the duty of mathematicians to define a notion of integral such that (2) has a meaning; the c_n are called the Fourier coefficients. Then, the series in (1) is defined in a formal way; it is called the Fourier series of f. Again, it is the duty of mathematicians to say how to obtain f from its Fourier series: summability methods and convergence in function spaces fit this purpose exactly. Actually Fourier series were one of the sources of functional analysis.

Fourier had predecessors. Daniel Bernoulli had the idea of expressing the solution of the problem of vibrating strings with the help of trigonometric series – that is, to express the motion of the string as the superposition of motions corresponding to pure harmonics. Euler applied formula (2) in particular cases. But, as Riemann observes, Fourier was the first to consider (1) and (2) as a whole: you analyse f through the Fourier formulas (2), you synthesize f through its Fourier series in (1). Analysis and synthesis are two complementary aspects of what is now called harmonic analysis.

Moreover, the motivation of Fourier was not the theory of vibrating strings – where harmonics introduce themselves in a rather natural way. It was the theory of heat. The Fourier series is only a part of a large project: to understand and predict heat diffusion. Fourier had to build a mathematical model for the propagation of heat. It is the so-called heat equation, or Fourier equation. He had to show how to use it in a number of particular cases. Fourier series are a practical tool for the computation of the temperature at a given point, in a given body, with given boundary conditions. This will be explained by Fourier himself when we quote him in this book.

Therefore the heritage of Fourier is not only the collection of problems, results, theories and concepts coming from formulas (1) and (2). It is more their *raison d'être*. Fourier addressed an important problem of nature. He was able to construct a good mathematical model. Then he wanted a general and powerful method to solve a type of equation: it was formulas (1) and (2). The tool being discovered, he tried to point out its actual extension as well as its precision in order to perform numerical computations. Mathematical modeling and algorithmic mathematics are part of the heritage of Fourier.

In particular, other connections with physics and numerical methods will be mentioned in this book. The historical part goes from Fourier to fast Fourier transform. However the unity of the book – expressed in its title – is the correspondence between the historical part and a self-contained mathematical exposition of the quite recent theory of wavelets. Wavelets are linked with physics and engineering; they are now the most general and powerful method for a scaled harmonic analysis needed in many ways, inside and outside mathematics. Of course, analysis and synthesis will appear in different forms, but the trace of formulas (1) and (2) will be quite visible.

It is time to say that formulas (1) and (2) were never written by Fourier. Complex exponentials were not used in Fourier series until well into the twentieth century. Therefore, in the historical part, (1) and (2) may appear in the form

$$f(x) = \frac{a_0}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx), \tag{3}$$

$$\begin{cases} a_n = \dfrac{1}{\pi} \displaystyle\int f(x) \cos nx \, dx, \\ b_n = \dfrac{1}{\pi} \displaystyle\int f(x) \sin nx \, dx, \end{cases} \tag{4}$$

or any equivalent form.

Chapter 1

WHO WAS FOURIER ?

Fourier is a pure product of the French Revolution. His life is a very interesting cross–section of French history in the years 1770–1830.

Joseph Fourier was born on 21 March 1768 in Auxerre, on the Yonne, in a rather poor family. His grandparents were peasants, his father a tailor, with 3 children from a first marriage and 13 (among them, Joseph) from a second. Both his father and mother died when he was very young. At the age of 10 years he was an orphan, but already noticed as a bright boy. He was taught French and Latin by the organist of the cathedral, then sent to the Military College of Auxerre.

The military colleges had just been created in twelve small towns in order to replace the Military School in Paris, founded in 1760. Among them, Beaumont and Brienne are known now because Laplace and Bonaparte studied there. In Auxerre the college was run by Benedictine monks, and the principles of education were quite progressive: to mix children of different classes, to practise living languages, German in particular, as well as Latin, and to be familiar with the use of geographical maps. Corporal punishment was forbidden but there was rather hard physical training. Joseph Fourier was happy and successful. He read Latin and occasionally wrote sermons (Arago says that some of them were delivered by high Church officials in Paris who did not bother writing sermons themselves). He discovered mathematics in the books of Bezout and Clairaut, worked day and night, graduated from college at 14, and was appointed as a teacher in the same college at 16. It was a bright record, but not yet a stable position.

For the college graduates there were two possible paths: Army or Church. Fourier chose to serve in the artillery, the most scientific arm. He was backed by Legendre, who already knew the interest and first works of Fourier on the localization of roots of an algebraic equation. But Fourier was poor and of poor origin. The minister of war answered Legendre : *"Fourier n'étant pas noble ne pourrait entrer dans l'artillerie, quand même il serait un second Newton"* (were he even a second Newton, Fourier could not enter artillery since he is not a noble). Fourier had to choose the Church.

He entered the Benedictine abbey of Saint--Benoit at Fleury, near Orleans, as a novice and mathematics teacher. From 1787 to 1789 he stayed there in complete isolation. Complete isolation is what his letters express. However the circumstances of his departure from Fleury suggest that he had some connection with the external world. He was supposed to take his vows on 5 November 1789. On 2 November the National Assembly ordered a suspension of religious vows. Fourier learned of this and, though already known as *"abbé Fourier"*, decided to give up the vows *"par respect pour les décrets de l'Assemblée Nationale"* (in consideration of the decrees of the National Assembly), and to leave Fleury.

He settled back in the military school at Auxerre as a humanities teacher. Already on 9 December 1789, he had presented a memoir on algebraic equations at the Academy of Sciences and the work was refereed by Monge, Legendre and Cousin. He was not involved in public affairs until 1793. This is the year that begins with the execution of Louis XVI and goes on to the European coalition against the French Republic, the uprisings in Vendée and many other regions, the mass levy of 300 000 men, and the revolutionary committees. In this most troubled time Fourier got involved and proved active and efficient by taking part in the revolutionary committee in Auxerre, recruiting volunteers in Burgundy, organizing food and military supplies in Orléans, moderating excesses in many places. In a letter written some two years later he explains his position:

> *"à mesure que les idées naturelles d'égalité se développèrent on a pu concevoir l'espérance sublime d'établir parmi nous un gouvernement libre exempt de rois et de prêtres, et d'affranchir de ce double joug la terre d'Europe depuis si longtemps usurpée. Je me passionnai aisément pour cette cause, qui est selon moi la plus grande et la plus belle qu'aucune nation ait jamais entreprise".*
>
> (as the natural ideas of equality were developing it became possible to conceive the sublime hope to establish among us a free government and to deliver the European soil from the kings and priests who have usurped it for so long. I became passionately fond of this cause, which I consider as the greatest and the best ever attempted by any nation.)

Fourier's mission to Orléans, a very troubled city, proved very successful as far as food and military supplies were concerned. However – maybe therefore – Fourier, who was on the *"sans–culottes"* side, was denounced to the *Convention*, arrested, liberated, arrested again, and once more liberated after the fall of Robespierre in July 1794. He returned to Auxerre and chose a newly created position: *"instituteur salarié par la nation"* ("State paid teacher" is not a good translation, since *nation* here evokes people as well as structure).

Then a new life began. In October 1794 the *Ecole normale* was created. The pupils were selected on a local basis, taking into account both involvement in teaching and devotion to revolutionary ideals. Fourier, *"professeur de physique et d'éloquence au collège national établi à Auxerre"*, was chosen and sent to Paris in December. There were 1 500 students and a handful of teachers: Lagrange, Laplace, Monge, Haüy, Berthollet, the greatest scientists of the time. The students were not well prepared (Fourier was an exception), the weather was cold, the amphitheatre (still existing, in the Museum of Natural History) was small, it was difficult to hear and to be heard, but the lectures were superb, and followed by debates between teachers and pupils. Traces exist. Not only were some lessons published, those of Laplace in particular, but stenographic records were taken – they waited until 1992 before being published (Dhombres 1992). For example, we see a discussion between Citizen Fourier and Citizen Monge on the definition of planes – Fourier, addressing Monge as *vous*, insisting that such a definition is necessary, as well as the definition of spheres given in Monge's course, and proposing to define planes using distances; Monge, addressing Fourier as *tu*, acknowledging Fourier's point of view but keeping his own, taking the plane as a primary notion. Obviously Fourier took advantage of the opportunities presented. However, for most students the level was too high. Winter cold, poor conditions, discrepancy between students and teachers resulted in the collapse of the school. The *Ecole normale* was recreated later on a more selective basis.

In the meantime the *Convention* founded the *Ecole polytechnique* as a science and military school. Fourier, highly appreciated by students and colleagues, taught there from 1795 to 1798 on several topics: differential calculus, integral calculus, statics, dynamics, hydrostatics and probabilities.

The first paper that he published, *"Mémoire sur la statique..."*, was printed in *Journal de l'Ecole Polytechnique* in 1798. His courses at the *Ecole Polytechnique* also contained original results, such as his theory of localization of real roots of an algebraic equation.

The scene changed again when Napoléon Bonaparte, who was about the same age as Fourier, led the French expedition to Egypt. Bonaparte had been elected a member of the *Institut de France*, probably because he brought back from Italy some geometrical constructions where ruler and compass are replaced by compass alone. He was proud of this membership and made use of it. In Egypt he signed his orders *"le membre de l'Institut, commandant l'armée d'Orient"*. He founded a copy of the French Institute in Cairo. Gaspard Monge was elected as President of this *Institut d'Egypte* and Fourier as *Secrétaire perpétuel.* Bonaparte left Egypt one year later to become First Consul in Paris, then emperor. Monge also went back to Paris. Fourier had increasing duties. He organized the activities of *Institut d'Egypte*, edited the proceedings in the journal *La Décade égyptienne*, wrote research papers on a great variety of subjects, from oases to the theory of equations, presented obituaries of the French generals Kléber and Desaix, wrote articles for *Le Courrier de l'Egypte*, directed a scientific expedition in Upper Egypt investigating the monuments and inscriptions, and conducted diplomatic negotiations, first with Mourad–Bey through his beautiful and celebrated wife Sitty Néfiçah, then with the English forces when the French had to withdraw from Egypt. The information he gathered was collected later in a monumental work called *Description de l'Egypte*, published in 1809, for which Fourier wrote an extensive *Préface historique.* Fourier's contributions qualified him as an important Egyptologist.

After his return to France, at the end of 1801, Napoléon gave him an important position as *Préfet de l'Isère.* It could have been the end of his scientific activities, but it was not. Not only did he write thousands of pages on Egypt, but also his main mathematical work, on the theory of heat, when he was prefect in Grenoble. Moreover, he was a good politician and administrator, supervising schools, mines, roads, health and agriculture. One of his achievements was to effect the long–wanted draining of the marshes in Bourgoin: political power was not sufficient, scientific competence and diplomatic talent were needed. He was clever enough to have good collaborators, to make friends in Grenoble (among them the Champollion family, whom he introduced to archeology), and to save time for his own scientific work.

Before Fourier, the theory of heat was not a very clear matter. The French Academy had proposed a competition in 1736 on the theme: *étude de la nature et de la propagation du feu*, meaning the nature and propagation of heat. But all candidates, including Euler and Voltaire, misunderstood the question and treated how fires develop. At the end of the 18th century there were debates on the nature of *"calorique"*, the term for heat in learned French. Is it a substance like a chemical element? The approach of Fourier is quite new: whatever may be the nature of heat, here is the way it should evolve. He does not speak of *calorique* any more: he writes