



VideoText *Interactive*

ALGEBRA

A COMPLETE COURSE

W O R K T E X T



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Unit I – The Structure of Mathematics

Part A – Mathematics as a Language

LESSON 1 Mathematical Parts of Speech

Objective: To identify the five parts of mathematical speech and understand their use.

Important Terms:

Number Symbols – more commonly called *numerals*, symbols used to represent quantities.

Operation Symbols – symbols such as + (add), – (subtract), \times (multiply) and \div (divide), used to indicate *action*.

Relation Symbols – symbols such as = (is equal to), $>$ (is greater than), and $<$ (is less than), used to show *comparisons*, as well as *combinations* such as \geq (is greater than or equal to), \leq (is less than or equal to), and the negations of all of these (\neq , \nless , \nless , \neq), among others.

Grouping Symbols – symbols such as () (parentheses), [] (brackets), and { } (braces), used to show *groupings*.

Placeholder Symbols – more commonly called *variables*, symbols such as a , b , c (letters of the alphabet) and \square (empty boxes), used to hold the place of a number until the number has been identified.

Example 1: Tell which of the following are relation symbols:

a. $17\frac{1}{2}$

b. \neq

c. $>$

d. \div

Solution: a. $17\frac{1}{2}$ is not a relation symbol. It is a number symbol meaning seventeen and one-half.

b. \neq is a relation symbol meaning “is *not* equal to.”

c. $>$ is a relation symbol meaning “is greater than.”

d. \div is not a relation symbol. It is an operation symbol meaning “divide.”

Lesson 1 – Exercises:

Tell what part of mathematical speech each of the following is, and state in words what each means.

- | | | | |
|--------------|-------------|----------------------|---------------|
| 1. c | 2. $[]$ | 3. $3.\overline{45}$ | 4. $+$ |
| 5. $=$ | 6. 63,231 | 7. \geq | 8. $\sqrt{5}$ |
| 9. $-$ | 10. $\{ \}$ | 11. 10^4 | 12. \div |
| 13. \neq | 14. $()$ | 15. $\frac{2}{3}$ | 16. 41.765 |
| 17. \times | 18. n | 19. \neq | 20. |

Part A – Mathematics as a Language

LESSON 2 **Mathematical Expressions**

Objective: To identify the four types of mathematical expressions and understand what to do with each.

Important Terms:

Closed Phrase – any mathematical expression which contains neither a relation symbol nor a placeholder symbol. For example, $2(5 - 1)$ is a closed phrase.

Open Phrase – any mathematical expression which contains a placeholder symbol but does not contain a relation symbol. For example, $6 + [3 - n]$ is an open phrase.

Closed Sentence – any mathematical expression which contains a relation symbol but does not contain a placeholder symbol. For example, $7(4 - 2) = 13$ is a closed sentence.

Open Sentence – any mathematical expression which contains both a relation symbol and a placeholder symbol. For example, $9 + 2[n - 4] > 12$ is an open sentence.

Example 1: Identify the following mathematical expressions by looking for relation symbols and placeholder symbols.

a. $17 - x$ b. $16 - (2 \cdot 5) \geq 6$ c. $\frac{12+3}{2}$ d. $20 = 4(n - 1)$

- Solution:**
- a. **Open Phrase** – Because there is no relation symbol, this expression is just a phrase. But there is a placeholder symbol, so the expression is open.
 - b. **Closed Sentence** – Because there is a relation symbol, this expression is a sentence. However, there is no placeholder symbol, so the expression is closed.
 - c. **Closed Phrase** – Because there is no relation symbol, this expression is just a phrase. Neither is there a placeholder symbol, so the expression is closed.
 - d. **Open Sentence** – There is a relation symbol, so the expression is a sentence. There is also a placeholder symbol, so the expression is open.

Example 2: Take the appropriate action for each of the following expressions. For open phrases, use a domain of $\{0, 1, 2\}$. For open sentences, use a replacement set of $\{4, 5, 6\}$.

a. $17 - x$ b. $16 - (2 \cdot 5) \geq 6$ c. $\frac{12+3}{2}$ d. $20 = 4(n - 1)$

- Solution:**
- a. Since this is an open phrase, the appropriate action is substitution from a domain and evaluation to a range.

Using $\{0, 1, 2\}$, we substitute and evaluate:

$$17 - (0) \rightarrow 17$$

$$17 - (1) \rightarrow 16$$

$$17 - (2) \rightarrow 15$$

So the range is $\{17, 16, 15\}$.

Example 2 cont'd:

- b. This expression is a closed sentence, so what we can do is tell if it is true or false.

$$16 - (2 \cdot 5) \geq 6$$

$$16 - 10 \geq 6$$

$$6 \geq 6$$

Since $6 = 6$ and that is one acceptable condition, the expression is true.

- c. Since this is a closed phrase, what we can do is evaluate it.

$$\frac{12 + 3}{2} = \frac{15}{2} = 7\frac{1}{2}$$

Its value is $7\frac{1}{2}$.

- d. This is an open sentence, so the appropriate action is to substitute from a replacement set and obtain a solution set of values that make the expression true. Using $\{4, 5, 6\}$ we substitute and determine truth or falsehood.

$$20 = 4([4] - 1)$$

$$= 4(3)$$

$$= 12$$

FALSE

$$20 = 4([5] - 1)$$

$$= 4(4)$$

$$= 16$$

FALSE

$$20 = 4([6] - 1)$$

$$= 4(5)$$

$$= 20$$

TRUE

So our solution set is $\{6\}$.

Lesson 2 – Exercises:

Tell whether each of the following expressions is an open phrase, closed phrase, open sentence, or closed sentence.

1. $6 + 8(3)$

2. $m + 2$

3. $5 - z = 2$

4. $8 - 2^3 = 4 - 2$

5. $\frac{12 + 8}{5} > 2 \cdot 2$

6. $w^2 - w$

7. $9y$

8. $\frac{15 + 18}{3} \neq 5 + 6$

9. $3k + 11 < 17$

10. $2 \cdot 4 - 1$

In the following exercises, take the appropriate mathematical action with each expression. For open phrases, use a domain of $\{0, 1, 2\}$. For open sentences, use a replacement set of $\{4, 5, 6\}$.

1. $6 + 8(3)$

2. $m + 2$

3. $5 - z = 2$

4. $8 - 2^3 = 4 - 2$

5. $\frac{12+8}{5} > 2 \cdot 2$

6. $w^2 - w$

7. $9y$

8. $\frac{15+18}{3} \neq 5 + 6$

9. $3k + 11 < 17$

10. $2 \cdot 4 - 1$

Part A – Mathematics as a Language

LESSON 3 Translation of Mathematical Symbols

Objective: To translate English expressions into expressions with mathematical symbols.

Example 1: Translate the following English phrases into phrases with mathematical symbols.

a. The sum of m and 9 decreased by the product of 6 and y .

b. The quotient of the cube of x and the square of z .

Solution: a. $\underbrace{\text{the sum of } m \text{ and } 9}_{(m+9)} \underbrace{\text{decreased by}}_{-} \underbrace{\text{the product of 6 and } y}_{6 \cdot y}$

So $(m + 9) - 6y$ is the desired phrase.

b. $\underbrace{\text{the quotient of}}_{\div} \underbrace{\text{the cube of } x}_{x^3} \underbrace{\text{and}}_{\div} \underbrace{\text{the square of } z}_{z^2}$

So $x^3 \div z^2$ or $\frac{x^3}{z^2}$ is the desired phrase.

Example 2: Translate the following English sentences into sentences with mathematical symbols.

- a. Six less than twice a number is 14.
- b. 58 is greater than the sum of 5 times a number and 18.

Solution: a. $\underbrace{\text{six less than}}_{-6} \underbrace{\text{twice a number}}_{2n} \text{ is } \underbrace{14}_{14}$

So $2n - 6 = 14$ is the desired sentence.

Note that “six less than” means “subtract 6,” so this must be placed to the **right** of $2n$.

b. $\underbrace{58}_{58} \underbrace{\text{is greater than}}_{>} \underbrace{\text{the sum of}}_{\text{the sum of}} \underbrace{5 \text{ times a number}}_{5m} \underbrace{\text{and}}_{+} \underbrace{18}_{18}$

So $58 > 5m + 18$ is the desired sentence.

Lesson 3 – Exercises:

Translate the following English phrases into phrases with mathematical symbols.

1. The sum of q and 6, increased by 4.
2. The product of m and n , divided by 11.
3. 27 more than the total of x and y .
4. The difference between p and 8.
5. The sum of a and b , divided by 6.
6. a to the power of 6, increased by 12.
7. The product of 10 and z , decreased by the product of 8 and x .
8. y multiplied by 6, increased by the product of 5 and z .
9. The cube of y , subtracted from the product of 5 and y .
10. The quantity x plus 5, times the quantity x minus 8.
11. The square of the difference of y and 9, divided by the product of 4 and t .
12. The difference between c and d , subtracted from the sum of q and r .
13. The sum of 6 and b , combined with the total of 7 and c .
14. The difference between the cube of v and the cube of w , divided by the square of e .
15. The square of h divided by b less than the cube of j .
16. 9 more than 4 times a number is equal to 55.

17. The result of 8 times a number decreased by 20 is the same as the sum of 4 times the number and 20.
18. Twice a number is 28.
19. 6 times a number decreased by 12 equals the sum of 3 times the number and 8.
20. 150 is the same as 6 times a number decreased by 30.
21. The difference between 25 and 4 times a number is less than 100 more than 3 times the number.
22. 15 is the difference between one-fourth of a number and 12.
23. Three times a number increased by 20 is greater than 56.
24. 14 more than 10 times a number is the same as the difference between 100 and 4 times the number.
25. The sum of 5 times a number and 30 is less than or equal to 50.

Unit I – The Structure of Mathematics

Part B – Further Investigation of Number Symbols

LESSON 1 The Development of Our Number System

Objective: To identify the different types of numbers and understand how they are related to each other.

Important Terms:

Well-Defined Operation – an operation which satisfies the conditions of *Existence* (must get an answer), *Uniqueness* (must get only one answer for each number combination), and *Closure* (answer must be in the set we are working in).

Natural Numbers – the numbers we use to count objects. The complete set of natural numbers is shown as $\{1, 2, 3, \dots\}$, usually called “the set N.” These numbers are also called counting numbers.

Whole Numbers – the natural numbers combined with the number 0. The complete set of whole numbers is shown as $\{0, 1, 2, 3, \dots\}$ and usually called “the set W.”

Integers – the whole numbers combined with all of their opposites. The complete set is shown as $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.