ALM 26
Advanced Lectures in Mathematics

Handbook of Moduli

(Volume Ⅲ)

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Editors: Gavril Farkas · Ian Morrison

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The Handbook of Moduli is dedicated to the memory of Eckart Viehweg, whose untimely death precluded a planned contribution, and to David Mumford, who first proposed the project, for all that they both did to nurture its subject; and to Angela Ortega and Jane Reynolds for everything that they do to sustain its editors.

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Volume III

Birational geometry for nilpotent orbits

Yoshinori Namikawa

Abstract. The following topics are discussed:

(1) Basic facts and examples of resolutions for nilpotent orbit.

(2) $\mathbb{Q}$-factorial terminalizations of nilpotent orbit closures and related birational geometry.

(3) Poisson deformations of nilpotent orbit closures.

Contents

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1. Introduction

The aim of this paper is to give an account of the birational point of view on nilpotent orbits in a complex simple Lie algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra and G the adjoint group. An adjoint orbit O in $\mathfrak{g}$ is called a nilpotent orbit if O consists of nilpotent elements of $\mathfrak{g}$. The closure $\bar{O}$ of O is then an affine variety with singularities. In general, $\bar{O}$ is not necessarily normal (see for example [15] in this direction). In this paper we shall take its normalization $\tilde{O}$ of $\bar{O}$ and consider the birational geometry on its (partial) resolutions. Each variety $\tilde{O}$ has symplectic singularities. More precisely, the smooth locus $\tilde{O}_{reg}$ admits the Kostant-Kirillov 2-form ω, which is d-closed and non-degenerate. Moreover, if we take a resolution $\mu : Y \to \tilde{O}$, then ω extends to a regular 2-form on Y. A resolution $\mu : Y \to \tilde{O}$ is called a crepant resolution if $K_Y = \mu^* K_{\tilde{O}}$. The nilpotent cone N is defined to be the subset of $\mathfrak{g}$ which consists of all nilpotent elements of $\mathfrak{g}$. By definition N is a disjoint union of all nilpotent orbits of $\mathfrak{g}$. There is a largest nilpotent orbit O_r and N coincides with its closure. Moreover, N is a normal variety. Let B be a Borel subgroup of G and let $T^*(G/B)$ be the cotangent bundle of the flag variety G/B. By using the Killing form of $\mathfrak{g}$, one can identify $T^*(G/B)$ with a vector bundle $G \times^B [\mathfrak{b}, \mathfrak{b}]$ over G/B. Then there is a natural map

$$\nu : G \times^B [\mathfrak{b}, \mathfrak{b}] \to \mathfrak{g}$$

defined by $[g, x] \to Ad_g(x)$. The image of ν coincides with N and ν gives a resolution of N ([25]). We call ν the *Springer resolution* of N. Since $T^*(G/B)$ admits a canonical symplectic 2-form and it coincides with the pull-back of the Kostant-Kirillov 2-form on O_r, the Springer resolution is a crepant resolution. One can generalize this construction to a parabolic subgroup Q of G. Let us start with the cotangent bundle $T^*(G/Q)$. Note that $T^*(G/Q)$ is identified with $G \times^Q \mathfrak{n}(\mathfrak{q})$ where $\mathfrak{n}(\mathfrak{q})$ is the nil-radical of $\mathfrak{q}$. In a similar way to the above, we have a map

$$\nu : T^*(G/Q) \to \mathfrak{g},$$

whose image is the closure of a nilpotent orbit O. In general, ν is not birational onto its image, but a generically finite projective morphism (see 2.6 for a non-birational Springer map). When ν gives a resolution of $\bar{O}$, we call ν the Springer resolution of $\bar{O}$. In this case, the Stein factorization

$$T^*(G/Q) \stackrel{\nu^n}{\to} \tilde{O} \to \bar{O}$$

gives a crepant resolution of $\tilde{O}$. B. Fu [7] proved the following.

Theorem ([7]). *Let O be a nilpotent orbit of $\mathfrak{g}$ and assume that $\tilde{O}$ admits a crepant resolution. Then it coincides with a Springer resolution. More exactly, there is a parabolic subgroup Q of G such that ν^n is the given crepant resolution.*

However there still remain interesting problems. At first, there actually exists a nilpotent orbit which has no crepant resolutions. Secondly, if $\tilde{O}$ has a crepant resolution, it is not unique, that is, the choice of Q is not unique even up to conjugacy class. Our purpose is to survey complete answers (cf. [18], [19] , [21] and [8]) to these problems.

A substitute for a crepant resolution is a $\mathbb{Q}$*-factorial terminalization*. A birational projective morphism $\mu : Y \to \tilde{O}$ is a $\mathbb{Q}$-factorial terminalization if Y has only $\mathbb{Q}$-factorial terminal singularities and $K_Y = \mu^* K_{\tilde{O}}$. The existence of a $\mathbb{Q}$-factorial terminalization is established by Birkar, Cascini, Hacon and M^{c}Kernan [2]. But, we shall give here more concrete forms of $\mathbb{Q}$-factorial terminalization. A hint is already in the work of Lusztig and Spaltenstein [17]. They introduced the notion of an *induced orbit*. Let us start with a parabolic subgroup Q of G and its Levi factor L(Q). Let $O' \subset \mathfrak{l}(\mathfrak{q})$ be a nilpotent orbit with respect to the adjoint L(Q)-action. Then one can make an associated bundle $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \bar{O}')$ and define a map

$$\nu : G \times^Q (\mathfrak{n}(\mathfrak{q}) + \bar{O}') \to \mathfrak{g}$$

by $\nu([g, x]) = \mathrm{Ad}_g(x)$. Since this is a G-equivariant closed map, its image is the closure of a nilpotent orbit O of $\mathfrak{g}$. Then we say that O is induced from O' and write $O = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}(\mathfrak{q})}(O')$. The map ν is called the generalized Springer map. The generalized Springer map ν is a generically finite projective morphism. But if ν is birational onto its image, then the Stein factorization

$$G \times^Q (\mathfrak{n}(\mathfrak{q}) + \bar{O}') \overset{\nu^n}{\to} \tilde{O} \to \bar{O}$$

gives a partial resolution of $\tilde{O}$. Now one can prove:

Theorem 2.6. *Let O be a nilpotent orbit of a complex simple Lie algebra* $\mathfrak{g}$*. Then there are a parabolic subalgebra* $\mathfrak{q}$ *of* $\mathfrak{g}$ *and a nilpotent orbit* O' *of* $\mathfrak{l}(\mathfrak{q})$ *such that the following holds:*

(1) $O = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}(\mathfrak{q})}(O')$.
(2) ν^n gives a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$.

In order to look for other $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$, we introduce a flat deformation of $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. For simplicity we put $\mathfrak{l} := \mathfrak{l}(\mathfrak{q})$ and let L be the corresponding Levi subgroup. Let $\mathfrak{r}(\mathfrak{q})$ be the solvable radical of $\mathfrak{q}$ and consider the variety $G \times^Q (\mathfrak{r}(\mathfrak{q}) + \bar{O}')$. Its normalization $X_{\mathfrak{q},O'}$ is isomorphic to $G \times^Q (\mathfrak{r}(\mathfrak{q}) + \tilde{O}')$. Let $\mathfrak{k}$ be the center of $\mathfrak{l}$. In 3.3 we shall define a map

$$X_{\mathfrak{q},O'} \to \mathfrak{k}$$

whose central fiber $X_{\mathfrak{q},O',0}$ is $G \times^Q (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. This map factorizes as

$$X_{\mathfrak{q},O'} \overset{\mu_{\mathfrak{q}}}{\to} \mathrm{Spec}\, \Gamma(X_{\mathfrak{q},O'}, \mathcal{O}_{X_{\mathfrak{q},O'}}) \to \mathfrak{k}.$$

Put

$$Y_{\mathfrak{l},O'} := \mathrm{Spec}\, \Gamma(X_{\mathfrak{q},O'}, \mathcal{O}_{X_{\mathfrak{q},O'}}).$$

An important fact is that $Y_{\mathfrak{l},O'}$ depends only on $\mathfrak{l}$ and O'. Moreover its central fiber $Y_{\mathfrak{l},O',0}$ is isomorphic to $\tilde{O}$. Define

$$\mathcal{S}(\mathfrak{l}) := \{\text{parabolic subalgebras } \mathfrak{q}' \text{ of } \mathfrak{g};\ \mathfrak{l}(\mathfrak{q}') = \mathfrak{l}\}.$$