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Advanced Lectures in Mathematics

Handbook of Moduli

(Volume III)

模手册 (卷Ⅲ)

Editors: Gavril Farkas · Ian Morrison



高等教育出版社
HIGHER EDUCATION PRESS

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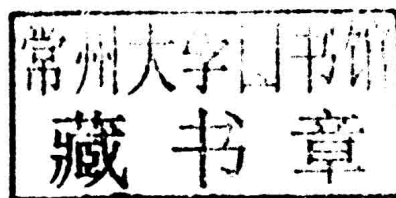
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4 Dewai Dajie, Beijing 100120, P. R. China, and
International Press
387 Somerville Ave, Somerville, MA, U. S. A.

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图书在版编目(CIP)数据

模手册. 3 = Handbook of Moduli. Vol. III : 英文 /
(德)法卡斯 (Farkas, G.), (美)莫里森 (Morrison, I.) 编.
— 北京 : 高等教育出版社, 2013. 1
ISBN 978-7-04-035174-3

I. ①模… II. ①法… ②莫… III. ①代数几何-文集
-英文 IV. ①O187-53

中国版本图书馆CIP数据核字(2012)第 242457号

策划编辑 王丽萍 责任编辑 王丽萍 封面设计 张申申 责任印制 朱学忠

出版发行	高等教育出版社	咨询电话	400-810-0598
社 址	北京市西城区德外大街 4 号	网 址	http://www.hep.edu.cn
邮政编码	100120		http://www.hep.com.cn
印 刷	涿州市星河印刷有限公司	网上订购	http://www.landaco.com
开 本	787mm×1092mm 1/16		http://www.landaco.com.cn
印 张	37.25	版 次	2013 年 1 月第 1 版
字 数	710 千字	印 次	2013 年 1 月第 1 次印刷
购书热线	010-58581118	定 价	128.00 元

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The Handbook of Moduli is dedicated to the memory of Eckart Viehweg, whose untimely death precluded a planned contribution, and to David Mumford, who first proposed the project, for all that they both did to nurture its subject; and to Angela Ortega and Jane Reynolds for everything that they do to sustain its editors.

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Birational geometry for nilpotent orbits

Yoshinori Namikawa

Abstract. The following topics are discussed:

- (1) Basic facts and examples of resolutions for nilpotent orbit.
- (2) \mathbb{Q} -factorial terminalizations of nilpotent orbit closures and related birational geometry.
- (3) Poisson deformations of nilpotent orbit closures.

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1. Introduction

The aim of this paper is to give an account of the birational point of view on nilpotent orbits in a complex simple Lie algebra. Let \mathfrak{g} be a complex simple Lie algebra and G the adjoint group. An adjoint orbit O in \mathfrak{g} is called a nilpotent orbit if O consists of nilpotent elements of \mathfrak{g} . The closure \bar{O} of O is then an affine variety with singularities. In general, \bar{O} is not necessarily normal (see for example [15] in this direction). In this paper we shall take its normalization \tilde{O} of \bar{O} and consider the birational geometry on its (partial) resolutions. Each variety \tilde{O} has symplectic singularities. More precisely, the smooth locus \tilde{O}_{reg} admits the Kostant-Kirillov 2-form ω , which is d-closed and non-degenerate. Moreover, if we take a resolution $\mu : Y \rightarrow \tilde{O}$, then ω extends to a regular 2-form on Y . A resolution $\mu : Y \rightarrow \tilde{O}$ is called a crepant resolution if $K_Y = \mu^* K_{\tilde{O}}$. The nilpotent cone N is defined to be the subset of \mathfrak{g} which consists of all nilpotent elements of \mathfrak{g} . By definition N is a disjoint union of all nilpotent orbits of \mathfrak{g} . There is a largest nilpotent orbit O_τ and N coincides with its closure. Moreover, N is a normal variety. Let B be a Borel subgroup of G and let $T^*(G/B)$ be the cotangent bundle of the flag variety G/B . By using the Killing form of \mathfrak{g} , one can identify $T^*(G/B)$ with a vector bundle $G \times^B [\mathfrak{b}, \mathfrak{b}]$ over G/B . Then there is a natural map

$$\nu : G \times^B [\mathfrak{b}, \mathfrak{b}] \rightarrow \mathfrak{g}$$

defined by $[g, x] \rightarrow \text{Ad}_g(x)$. The image of ν coincides with N and ν gives a resolution of N ([25]). We call ν the *Springer resolution* of N . Since $T^*(G/B)$ admits a canonical symplectic 2-form and it coincides with the pull-back of the Kostant-Kirillov 2-form on O_τ , the Springer resolution is a crepant resolution. One can generalize this construction to a parabolic subgroup Q of G . Let us start with the cotangent bundle $T^*(G/Q)$. Note that $T^*(G/Q)$ is identified with $G \times^Q \mathfrak{n}(q)$ where $\mathfrak{n}(q)$ is the nil-radical of q . In a similar way to the above, we have a map

$$\nu : T^*(G/Q) \rightarrow \mathfrak{g},$$

whose image is the closure of a nilpotent orbit O . In general, ν is not birational onto its image, but a generically finite projective morphism (see 2.6 for a non-birational Springer map). When ν gives a resolution of \bar{O} , we call ν the Springer resolution of \bar{O} . In this case, the Stein factorization

$$T^*(G/Q) \xrightarrow{\nu^n} \tilde{O} \rightarrow \bar{O}$$

gives a crepant resolution of \bar{O} . B. Fu [7] proved the following.

Theorem ([7]). *Let O be a nilpotent orbit of \mathfrak{g} and assume that \tilde{O} admits a crepant resolution. Then it coincides with a Springer resolution. More exactly, there is a parabolic subgroup Q of G such that ν^n is the given crepant resolution.*

However there still remain interesting problems. At first, there actually exists a nilpotent orbit which has no crepant resolutions. Secondly, if \bar{O} has a crepant

resolution, it is not unique, that is, the choice of Q is not unique even up to conjugacy class. Our purpose is to survey complete answers (cf. [18], [19], [21] and [8]) to these problems.

A substitute for a crepant resolution is a \mathbb{Q} -factorial terminalization. A birational projective morphism $\mu : Y \rightarrow \tilde{O}$ is a \mathbb{Q} -factorial terminalization if Y has only \mathbb{Q} -factorial terminal singularities and $K_Y = \mu^* K_{\tilde{O}}$. The existence of a \mathbb{Q} -factorial terminalization is established by Birkar, Cascini, Hacon and McKernan [2]. But, we shall give here more concrete forms of \mathbb{Q} -factorial terminalization. A hint is already in the work of Lusztig and Spaltenstein [17]. They introduced the notion of an *induced orbit*. Let us start with a parabolic subgroup Q of G and its Levi factor $L(Q)$. Let $O' \subset l(q)$ be a nilpotent orbit with respect to the adjoint $L(Q)$ -action. Then one can make an associated bundle $G \times^Q (n(q) + \tilde{O}')$ and define a map

$$\nu : G \times^Q (n(q) + \tilde{O}') \rightarrow \mathfrak{g}$$

by $\nu([g, x]) = \text{Ad}_g(x)$. Since this is a G -equivariant closed map, its image is the closure of a nilpotent orbit O of \mathfrak{g} . Then we say that O is induced from O' and write $O = \text{Ind}_{l(q)}^{\mathfrak{g}}(O')$. The map ν is called the generalized Springer map. The generalized Springer map ν is a generically finite projective morphism. But if ν is birational onto its image, then the Stein factorization

$$G \times^Q (n(q) + \tilde{O}') \xrightarrow{\nu^n} \tilde{O} \rightarrow \tilde{O}$$

gives a partial resolution of \tilde{O} . Now one can prove:

Theorem 2.6. *Let O be a nilpotent orbit of a complex simple Lie algebra \mathfrak{g} . Then there are a parabolic subalgebra q of \mathfrak{g} and a nilpotent orbit O' of $l(q)$ such that the following holds:*

- (1) $O = \text{Ind}_{l(q)}^{\mathfrak{g}}(O')$.
- (2) ν^n gives a \mathbb{Q} -factorial terminalization of \tilde{O} .

In order to look for other \mathbb{Q} -factorial terminalizations of \tilde{O} , we introduce a flat deformation of $G \times^Q (n(q) + \tilde{O}')$. For simplicity we put $l := l(q)$ and let L be the corresponding Levi subgroup. Let $\tau(q)$ be the solvable radical of q and consider the variety $G \times^Q (\tau(q) + \tilde{O}')$. Its normalization $X_{q, O'}$ is isomorphic to $G \times^Q (\tau(q) + \tilde{O}')$. Let \mathfrak{k} be the center of l . In 3.3 we shall define a map

$$X_{q, O'} \rightarrow \mathfrak{k}$$

whose central fiber $X_{q, O', 0}$ is $G \times^Q (n(q) + \tilde{O}')$. This map factorizes as

$$X_{q, O'} \xrightarrow{\mu_q} \text{Spec } \Gamma(X_{q, O'}, \mathcal{O}_{X_{q, O'}}) \rightarrow \mathfrak{k}.$$

Put

$$Y_{l, O'} := \text{Spec } \Gamma(X_{q, O'}, \mathcal{O}_{X_{q, O'}}).$$

An important fact is that $Y_{l, O'}$ depends only on l and O' . Moreover its central fiber $Y_{l, O', 0}$ is isomorphic to \tilde{O} . Define

$$\mathcal{S}(l) := \{\text{parabolic subalgebras } q' \text{ of } \mathfrak{g}; l(q') = l\}.$$