Undergraduate Texts in Mathematics

Wendell Fleming

Functions of Several Variables

Second Edition

多元函数 第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

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With 96 Illustrations

Springer

图书在版编目（CIP）数据

多元函数：第2版 = Functions of Several Variables 2nd ed.：英文/（美）弗莱明（Fleming，W.）著. —影印本. —北京：世界图书出版公司北京公司，2010. 9

ISBN 978-7-5100-2730-7

Ⅰ. ①多… Ⅱ. ①弗… Ⅲ. ①多变量—函数—英文 Ⅳ. ①O13②O174

中国版本图书馆 CIP 数据核字（2010）第 168584 号

书　　名：Functions of Several Variables 2nd ed.
作　　者：Wendell Fleming

中 译 名：多元函数 第2版
责任编辑：高蓉　刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街137号　100010）
联系电话：010-64021602，010-64015659
电子信箱：kjb@wpcbj.com.cn

开　　本：24开
印　　张：18
版　　次：2010年09月
版权登记：图字：01-2010-1415

书　　号：978-7-5100-2730-7/O·839　　定　　价：45.00元

(continued after index)

Undergraduate Texts in Mathematics

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S. Axler
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Springer

New York
Berlin
Heidelberg
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Santa Clara
Singapore
Tokyo

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Mathematics Subject Classification (1991): 26–01, 28–01, 58–01

Library of Congress Cataloging in Publication Data
Fleming, Wendell Helms, 1928–
Functions of several variables.
(Undergraduate texts in mathematics)
Bibliography: p.
Includes index.
1. Functions of several real variables.
I. Title
QA331.F63 1977 515′.84 76-40029

ISBN 0-387-90206-6 Springer-Verlag New York Heidelberg Berlin
ISBN 3-540-90206-6 Springer-Verlag Berlin Heidelberg New York SPIN 10570374

Preface

The purpose of this book is to give a systematic development of differential and integral calculus for functions of several variables. The traditional topics from advanced calculus are included: maxima and minima, chain rule, implicit function theorem, multiple integrals, divergence and Stokes's theorems, and so on. However, the treatment differs in several important respects from the traditional one. Vector notation is used throughout, and the distinction is maintained between n-dimensional euclidean space E^n and its dual. The elements of the Lebesgue theory of integrals are given. In place of the traditional vector analysis in E^3, we introduce exterior algebra and the calculus of exterior differential forms. The formulas of vector analysis then become special cases of formulas about differential forms and integrals over manifolds lying in E^n.

The book is suitable for a one-year course at the advanced undergraduate level. By omitting certain chapters, a one semester course can be based on it. For instance, if the students already have a good knowledge of partial differentiation and the elementary topology of E^n, then substantial parts of Chapters 4, 5, 7, and 8 can be covered in a semester. Some knowledge of linear algebra is presumed. However, results from linear algebra are reviewed as needed (in some cases without proof).

A number of changes have been made in the first edition. Many of these were suggested by classroom experience. A new Chapter 2 on elementary topology has been added. Additional physical applications—to thermodynamics and classical mechanics—have been added in Chapters 6 and 8. Different proofs, perhaps easier for the beginner, have been given for two main theorems (the Inverse Function Theorem and the Divergence Theorem.)

The author is indebted to many colleagues and students at Brown University for their valuable suggestions. Particular thanks are due Hildegarde Kneisel, Scott Shenker, and Joseph Silverman for their excellent help in preparing this edition.

Wendell H. Fleming

Providence, Rhode Island
June, 1976

Contents

Euclidean spaces 1

This book is concerned with the differential and integral calculus of functions of several variables. For this purpose one needs first to know some basic properties of euclidean space of arbitrary finite dimension n. We begin this chapter with a brief review of the real numbers and the elements of vector algebra and geometry of such spaces. Later in the chapter the concepts of neighborhood, open set, and closed set are introduced. These constitute the basis for studying what are called topological properties of n-dimensional space.

Format

The word "Theorem" has been reserved for what the author considers the most important results. Results of lesser depth or interest are labeled "Proposition." The symbol $\square$ indicates the end of the proof of a theorem or proposition. Occasionally part of a proof is left to the reader as a homework exercise. The sections marked with an asterisk (*) may be omitted without disrupting the organization. References are given at the end of the book.

We presume that the reader is acquainted with the most elementary aspects of set theory. The symbols

$$\in, \notin, \cup, \cap, -, \subset$$

stand, respectively, for *is an element of*, *is not an element of*, *union*, *intersection*, *difference*, and *inclusion*. Sets ordinarily are denoted by capital italicized letters. A set is described either by listing its elements or by some property characterizing them. Thus $\{2, 5, 7\}$ is the set whose elements are the three numbers 2, 5, and 7. If S is a set and π a property pertaining to elements of S, then $\{p \in S : \pi\}$ denotes *the set of all* $p \in S$ *with property* π. For example, if $Z = \{1, 2, \ldots\}$ is the set of natural numbers, then $S = \{x \in Z : x = 2y - 1$

for some $y \in Z\}$ is the set of odd natural numbers. The set $\{x \in Z : x^2 = 3\}$ is the empty set. The set $\{x \in Z : x(x-1) = x^2 - x\}$ is all of Z.

When the set S in question is clear from the context, we abbreviate by writing simply $\{p : \pi\}$.

1.1 The real number system

While calculus has been motivated in large part by problems from geometry and physics, its foundations rest upon the idea of number. Therefore a thorough treatment of calculus should begin with a study of the real numbers. The real number system satisfies axioms about arithmetic and order, which express properties of numbers with which everyone is familiar from elementary mathematics.

We list these properties as Axioms I and II.

Axiom I

(a) Any two real numbers have a sum $x + y$ and a product xy, which are also real numbers. Moreover,

Commutative law	$x + y = y + x,$	$xy = yx,$
Associative law	$x + (y + z) = (x + y) + z,$	$x(yz) = (xy)z,$
Distributive law	$x(y + z) = xy + xz$	

for every x, y, and z.

(b) There are two (distinct) real numbers 0 and 1, which are identity elements under addition and multiplication, respectively:

$$x + 0 = x, \qquad x1 = x$$

for every x.

(c) Every real number x has an inverse $-x$ with respect to addition, and if $x \neq 0$, an inverse x^{-1} with respect to multiplication:

$$x + (-x) = 0, \qquad xx^{-1} = 1.$$

Axiom II. There is a relation $<$ between real numbers such that:

(a) For every pair of numbers x and y, exactly one of the following alternatives holds: $x < y$, $x = y$, $y < x$.
(b) $w < x$ and $x < y$ imply $w < y$ (transitive law).
(c) $x < y$ implies $x + z < y + z$ for every z.
(d) $x < y$ implies $xz < yz$ whenever $0 < z$.

From Axioms I and II follow all of the ordinary laws of arithmetic. In algebra any set with two operations (usually called *addition* and *multiplication*) having the properties listed in Axiom I is called a *field*. A field is called *ordered* if there is in it a relation $<$ satisfying Axiom II.

The real numbers form an ordered field. However, this is by no means the only ordered field. For example, the rational numbers also form an ordered field. We recall that x rational means that $x = p/q$, where p and q are integers and $q \neq 0$. Yet another axiom is needed to characterize the real number system. This axiom can be introduced in several ways. Perhaps the simplest of these is Axiom III, stated below in terms of least upper bounds.

In Section 2.3 we state other axioms that turn out to be equivalent to Axiom III. We should warn the reader that Axiom III is more subtle than Axioms I and II, and that one becomes aware only gradually of its implications. However, this axiom is the foundation stone for some of the most important theorems in calculus.

Let S be a nonempty set of real numbers. If there is a number c such that $x \leq c$ for every $x \in S$, then c is called an *upper bound* for S. If c is an upper bound for S and $b \geq c$, then b is also an upper bound for S.

Axiom III. Any set S of real numbers that has an upper bound has a least upper bound.

The least upper bound for S is denoted by $\sup S$. If S has no upper bound, then we set $\sup S = +\infty$.

A number d is a lower bound for S if $d \leq x$ for every $x \in S$. If S has a lower bound, then (Problem 2) S has a greatest lower bound. It is denoted by $\inf S$. If S has no lower bound, then we set $\inf S = -\infty$.

Example 1. Let $S = \{1, 2, 3, \ldots\}$, the set of positive integers. Then $\sup S = +\infty$ and $\inf S = 1$.

Example 2. Let a and b be real numbers with $a < b$. The sets

$$[a, b] = \{x : a \leq x \leq b\}, \qquad (a, b) = \{x : a < x < b\},$$
$$[a, b) = \{x : a \leq x < b\}, \qquad (a, b] = \{x : a < x \leq b\}$$

are called *finite intervals* with endpoints a and b. The first of these intervals is called *closed*, the second *open*, and the last two, *half-open*. In each instance b is the least upper bound and a is the greatest lower bound.

In the same way, the *semiinfinite intervals*

$$[a, \infty) = \{x : x \geq a\}, \qquad (a, \infty) = \{x : x > a\}$$

are called closed and open, respectively, and have a as greatest lower bound. The corresponding intervals $(-\infty, b]$, $(-\infty, b)$ have b as least upper bound.

Let S be a set that has an upper bound. Example 2 shows that the number $\sup S$ need not belong to S. If $\sup S$ does happen to be an element of S, then it is the largest element of S and we write "$\max S$" instead of "$\sup S$." Similarly, if S is bounded below and $\inf S$ is an element of S, then we write for it "$\min S$."

EXAMPLE 3. Let $S = \{x : x^2 < 2 \text{ and } x \text{ is a rational number}\}$. Then $\sqrt{2} = \sup S$ and $-\sqrt{2} = \inf S$. Since $\sqrt{2}$ is not a rational number, this example shows that the least upper bound axiom would no longer hold if we replaced the real number system by the rational number system.

EXAMPLE 4. Let $S = \{\sin x : x \in [-\pi, \pi]\}$. Then $-1 = \min S$, $1 = \max S$.

The real number system also satisfies the *archimedean property*. This means that for every $\varepsilon > 0$, $x > 0$ there exists a positive integer m such that $x < m\varepsilon$. To prove it, suppose to the contrary that for some pair ε, x of positive numbers, $m\varepsilon \leq x$ for every $m = 1, 2, \ldots$. Then x is an upper bound for the set $S = \{\varepsilon, 2\varepsilon, 3\varepsilon, \ldots\}$. Let $c = \sup S$. Then $(m + 1)\varepsilon \leq c$ and therefore $m\varepsilon \leq c - \varepsilon$, for each $m = 1, 2, \ldots$. Hence $c - \varepsilon$ is an upper bound for S smaller than $\sup S$, a contradiction. This proves the archimedean property.

We shall not prove that there actually is a system satisfying Axioms I, II, and III. There are two well-known methods of constructing the real number system, starting from the rational numbers. One of them is the method of Dedekind cuts and the other is Cantor's method of Cauchy sequences.

Axioms I, II, and III characterize the real numbers; in other words, any two systems satisfying these three axioms are essentially the same. To put this more precisely in algebraic language, any two ordered fields satisfying Axiom III are isomorphic.

For proofs of these facts, refer to the book by Birkhoff and McLane [2, Chapter III].

PROBLEMS

1. Find the least upper bound $\sup S$ and greatest lower bound $\inf S$ of each of the following sets:
   (a) $\{x : x^2 - 3x + 2 < 0\}$.
   (b) $\{x : x^3 + x^2 - 2x \leq 2\}$.
   (c) $\{\sin x + \cos x : x \in [0, \pi]\}$.
   (d) $\{x \exp x : x < 0\}$. [*Note*: exp denotes the exponential function, $\exp x = e^x$, where e is the base for natural logarithms.]

   State whether $\sup S$ and $\inf S$ are elements of S.

2. Let $T = \{x : -x \in S\}$. Show that $-\sup T = \inf S$.

3. Let x and y be real numbers with $x < y$. Show that there is a rational number z such that $x < z < y$. [*Hint*: By the archimedean property there is a positive integer q such that $q^{-1} < y - x$. Let $z = p/q$, where p is the smallest positive integer such that $qx < p$.]

[*Note*: In this book, "Show that ..." and "Prove that ..." both mean "give a valid mathematical proof."]

1.2 Euclidean E^n

In this book we denote the real number system by E^1. Let us now define the space E^n, whose elements are n-tuples of real numbers. The elements of E^n will be called vectors.

Scalars and vectors

By *scalar* we mean a real number. In elementary mathematics a vector is described as a quantity that has both direction and length. Vectors are illustrated by drawing arrows issuing from a given point $\mathbf{0}$. The point at the head of the arrow specifies the vector. Therefore we may (and shall) say that this point *is* the vector. Thus in two dimensions a vector is just a point (x, y) of the plane E^2. Vectors in E^2 are added by the parallelogram law, which amounts to adding corresponding components. Thus

$$(x, y) + (u, v) = (x + u, y + v).$$

The product of (x, y) by a scalar c is the vector (cx, cy). The zero vector is $(0, 0)$.

With this in mind, let us define the space E^n for any positive integer n. The elements of E^n are n-tuples $(x^1, \ldots, x^n)$ of real numbers. For short, we write $\mathbf{x}$ for the n-tuple $(x^1, \ldots, x^n)$. The notation $\mathbf{x} \in E^n$ means "$\mathbf{x}$ is an element of E^n." The elements of E^n are called *vectors*, and also *points*, depending on which term seems more suggestive in the context. Addition and scalar multiplication are defined in E^n as follows. If

$$\mathbf{x} = (x^1, \ldots, x^n), \qquad \mathbf{y} = (y^1, \ldots, y^n)$$

are any two elements of E^n, then

$$\mathbf{x} + \mathbf{y} = (x^1 + y^1, \ldots, x^n + y^n).$$

If $\mathbf{x} \in E^n$ and c is a scalar, then

$$c\mathbf{x} = (cx^1, \ldots, cx^n).$$

The *zero element* of E^n is

$$\mathbf{0} = (0, \ldots, 0).$$

With these definitions E^n satisfies the axioms for a vector space (Appendix A.1). The term "vector" is reserved for elements of E^n rather than those of any space satisfying these axioms.

The superscripts should not be confused with powers of x. For instance, $(x^i)^2$ means the square of the ith entry x^i of the n-tuple $(x^1, \ldots, x^n)$.

If $n = 1$ we identify the 1-tuple $\mathbf{x} = (x)$ with the scalar x. In this case addition and scalar multiplication reduce to ordinary addition and multiplication of real numbers. If $n = 2$ or 3 we usually write (x, y) or (x, y, z), as is commonly done in elementary analytic geometry, rather than (x^1, x^2) or

(x^1, x^2, x^3). Practically all of the theorems are stated and proved for arbitrary dimension n. However, the special cases $n = 2, 3$ will frequently appear in the examples and homework problems.

The notions of vector sum and multiplication by scalars determine the vector-space structure of E^n, but are not enough to define the concepts of distance and angle. These arise by introducing an inner product in E^n. An inner product assigns to each pair $\mathbf{x}, \mathbf{y}$ of vectors a scalar, and must have the four properties listed in Problem 2 at the end of the section. The one which we shall use is the *euclidean inner product*, denoted by $\cdot$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x^i y^i.$$

The vector space E^n with this inner product is called *euclidean n-space*. Other inner products in E^n are considered in Section 2.11.

The *euclidean norm* (or *length*) of a vector $\mathbf{x}$ is

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}.$$

It is positive, except when $\mathbf{x} = \mathbf{0}$, and satisfies the following two important inequalities. For every $\mathbf{x}, \mathbf{y} \in E^n$,

(1.1) $\quad |\mathbf{x} \cdot \mathbf{y}| \le |\mathbf{x}||\mathbf{y}| \quad$ (Cauchy's inequality),

(1.2) $\quad |\mathbf{x} + \mathbf{y}| \le |\mathbf{x}| + |\mathbf{y}| \quad$ (triangle inequality).

PROOF OF (1.1). If $\mathbf{y} = \mathbf{0}$, then both sides of (1.1) are $\mathbf{0}$. Therefore let us suppose that $\mathbf{y} \ne \mathbf{0}$. For every scalar t,

$$(\mathbf{x} + t\mathbf{y}) \cdot (\mathbf{x} + t\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2t\mathbf{x} \cdot \mathbf{y} + t^2\mathbf{y} \cdot \mathbf{y},$$

since the inner product is commutative and distributive [Problem 2, parts (a), (b), and (c)]. The left-hand side is $|\mathbf{x} + t\mathbf{y}|^2$, and $\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$, $\mathbf{y} \cdot \mathbf{y} = |\mathbf{y}|^2$. The right-hand side is quadratic in t and has a minimum when

$$t = t_0 = -\frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}.$$

Substituting this expression for t, we find that

$$0 \le |\mathbf{x} + t_0\mathbf{y}|^2 = |\mathbf{x}|^2 - \frac{|\mathbf{x} \cdot \mathbf{y}|^2}{|\mathbf{y}|^2},$$

or

$$|\mathbf{x} \cdot \mathbf{y}|^2 \le |\mathbf{x}|^2|\mathbf{y}|^2.$$

The last inequality is equivalent to Cauchy's inequality. □

From the proof we see that equality in Cauchy's inequality is equivalent to the fact that $|\mathbf{x} + t_0\mathbf{y}| = 0$, that is, that $\mathbf{x} + t_0\mathbf{y} = \mathbf{0}$. Thus, if $\mathbf{y} \ne \mathbf{0}$, $|\mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}||\mathbf{y}|$ *if and only if* $\mathbf{x}$ *is a scalar multiple of* $\mathbf{y}$. If $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|$, then $\mathbf{x}$ is a *nonnegative* scalar multiple of $\mathbf{y}$ (and conversely).