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Quantum Mechanics of Non-Hamiltonian and Dissipative Systems

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Preface

This book is the expanded version of lectures on quantum mechanics, which author read for students of the graduate level and which have been published in Russian. The main attention is given to the consecutive and consistent description of foundations of modern quantum mechanics. Difference of the suggested book from others is consistent use of the functional analysis and operator algebras. To read the text, preliminary knowledge of these sections of mathematics is not required. All the necessary information, which is beyond usual courses of the mathematical analysis and linear algebra, is included.

To describe the theory, we use the fact that quantum and classical mechanics are connected not only by limiting transition, but also realized by identical mathematical structures. A common basis to formulate the theory is an assumption that classical and quantum mechanics are different representations of the same totality of mathematical structures, i.e., the so-called Dirac correspondence principle. For construction of quantum theory, we consider mathematical concepts that are the general for Hamiltonian and non-Hamiltonian systems. Quantum dynamics is described by the one-parameter semi-groups and the differential equations on operator spaces and algebras. The Lie–Jordan algebraic structure, Liouville space and superoperators are used. It allows not only to consistently formulate the evolution of quantum systems, but also to consider the dynamics of a wide class of quantum systems, such as the open, non-Hamiltonian, dissipative, and nonlinear systems. Hamiltonian systems in pure states are considered as special cases of quantum dynamical systems.

The closed, isolated and Hamiltonian systems are idealizations that are not observable and therefore do not exist in the real world. As a rule, any system is always embedded in some environment and therefore it is never really closed or isolated. Frequently, the relevant environment is in principle unobservable or is unknown. This would render the theory of non-Hamiltonian and dissipative quantum systems to a fundamental generalization of quantum mechanics. The quantum theory of Hamiltonian systems, unitary evolution, and pure states should be considered as special cases of the generalized approach.

Usually the quantum mechanics is considered as generalization of classical mechanics. In this book the quantum mechanics is formulated as a generalization of modern nonlinear dynamics of dissipative and non-Hamiltonian systems. The quantization of equations of motion for dissipative and non-Hamiltonian classical systems is formulated in this book. This quantization procedure allows one to derive quantum analogs of equations with regular and strange attractors. The regular attractors are considered as stationary states of non-Hamiltonian and dissipative quantum systems. In the book, the quantum analogs of the classical systems with strange attractors, such as Lorenz and Rössler systems, are suggested. In the text, the main attention is devoted to non-Hamiltonian and dissipative systems that have the wide possibility to demonstrate the complexity, chaos and self-organization.

The text is self-contained and can be used without introductory courses in quantum mechanics and modern mathematics. All the necessary information, which is beyond undergraduate courses of the mathematics, is presented in the book. Therefore this book can be used in the courses for graduate students. In the book the modern structure of the quantum theory and new fundamental results of last years are described. Some of these results are not considered in monographs and text books. Therefore the book is supposed to be useful for physicists and mathematicians who are interested in the modern quantum theory, nonlinear dynamics, quantization and chaos.

The book consists of two interconnected parts. The first part is devoted to the quantum kinematics that defines the properties of quantum observable, states and expectation values. In the second part, we consider the quantum dynamics that describes the time evolution of the observables and states.

Quantum mechanics has its mathematical language. It consists of the operator algebras, functional analysis, theory of one-parameter semi-groups and operator differential equations. Although we can have some sort of understanding of quantum mechanics without knowing its mathematical language, the precise and deep meaning of the physical notions cannot be obtained without using operator algebras, functional analysis, etc. Many theorems of operator algebra and functional analysis, etc. are easy to understand and use, although their proofs may be quite technical and time-consuming to present. Therefore we explain the meaning and significance of the theorems and ask reader to use them without proof.

The author is greatly indebted to Professor George M. Zaslavsky for his invaluable suggestions and comments. Thanks are expressed also to Edward E. Boos, Vyacheslav A. Ilin, Victor I. Savrin, Igor V. Volovich, colleagues of THEP division, and my family for their help and invaluable support during the work on the book. Finally, the author wishes to express his appreciation to Elsevier for the publication of this book.

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September 2007

Contents

A Very Few Preliminaries

To motivate the introduction of the basic concepts of the theory of non-Hamiltonian and dissipative systems, we begin with some definitions.

1. Potential and conservative systems

Suppose that a classical system, whose position is determined by a vector $\mathbf{x}$ in a region $\mathcal{M}$ of n-dimensional phase-space $\mathbb{R}^n$, moves in a field $\mathbf{F}(\mathbf{x})$. The motion of the system is described by the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}). \tag{1}$$

Let us give the basic definitions regarding this system.

(1) If the vector field $\mathbf{F}(\mathbf{x})$ satisfies the condition

$$curl\, \mathbf{F}(\mathbf{x}) = 0$$

for all $\mathbf{x} \in \mathcal{M}$, then the system is called *potential*, or *locally potential*. The field $\mathbf{F}(\mathbf{x})$ is called irrotational.

(2) If there is a unique single-valued function $H = H(\mathbf{x})$ for all $x \in \mathcal{M}$ such that

$$\mathbf{F}(\mathbf{x}) = grad\, H(\mathbf{x}),$$

then the system is *gradient*, or *globally potential*.

The globally potential system is locally potential. The converse statement does not hold in general. It is well known that a locally potential system with the field $\mathbf{F} = (-y/r^2)\mathbf{e}_1 + (x/r^2)\mathbf{e}_2$, where $r^2 = x^2 + y^2$ in the region $\mathcal{M} = \{(x, y) \in \mathbb{R}^2\colon\ (x, y) \neq (0, 0)\}$ is not globally potential.

(3) If there are $\mathbf{x} \in \mathcal{M} \subset \mathbb{R}^n$ such that

$$curl\, \mathbf{F}(\mathbf{x}) \neq 0,$$

then the system is called *nonpotential*.

(4) If we have the condition

$$div\, \mathbf{F}(\mathbf{x}) = 0$$

for all $\mathbf{x} \in \mathcal{M}$, then the system is called *nondissipative*. The vector field $\mathbf{F}(\mathbf{x})$ is called solenoidal.

2. Hamiltonian and non-Hamiltonian classical systems

Let $\mathcal{M}$ be a symplectic manifold and let $\mathbf{x} = (q, p)$.

(1) The locally potential system on $\mathcal{M}$ is called *locally Hamiltonian.*
(2) The globally potential system on $\mathcal{M}$ is called *globally Hamiltonian.*
(3) The nonpotential system on $\mathcal{M}$ is called *non-Hamiltonian.*
(4) If $div\,\mathbf{F}(\mathbf{x}) \neq 0$ for some $\mathbf{x} \in \mathcal{M}$, then the system is called *generalized dissipative.*

3. Examples of non-Hamiltonian systems

Suppose that a classical system, whose position and momentum are described by vectors $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$, moves in the force field $F(q, p) = (F_1, \ldots, F_n)$. The motion of the system is defined by the equations

$$\frac{dq_k}{dt} = \frac{\partial H(q, p)}{\partial p_k}, \qquad \frac{dp_k}{dt} = -\frac{\partial H(q, p)}{\partial q_k} + F_k(q, p). \tag{2}$$

The Hamiltonian function $H(q, p) = p^2/2m + U(q)$ gives the Newton's equations

$$\frac{d^2 q_k}{dt^2} = -\frac{\partial U(q)}{\partial q_k} + F_k(q, mv),$$

where $v = dq/dt$. If the conditions

$$\frac{\partial F_k(q, p)}{\partial p_l} = 0, \qquad \frac{\partial F_k(q, p)}{\partial q_l} - \frac{\partial F_l(q, p)}{\partial q_l} = 0 \tag{3}$$

hold for all q, p, then equations (2) describe a classical Hamiltonian system. If these conditions are not satisfied, then (2) is a non-Hamiltonian system. If

$$\Omega(q, p) = \sum_{k=1}^{n} \frac{\partial F_k(q, p)}{\partial p_k} \neq 0,$$

then we have a generalized dissipative system. For example, the force field

$$F_k(q, p) = \sum_{l=1}^{n} a_{kl} p_l + \sum_{l,s=1}^{n} b_{kls} p_l p_s \tag{4}$$

describes non-Hamiltonian system.

Suppose that $H(q, p) = p^2/2m$ and $F_k(q, p)$ is defined by (4). Using the variables $x = p_1$, $y = p_2$, $z = p_3$, we can obtain the well-known Lorenz and Rössler systems in the space of $(x, y, z) \in \mathbb{R}^3$. The field

$$F_1 = -\sigma x + \sigma y, \qquad F_2 = rx - y - xz, \qquad F_3 = -bz + xy,$$

gives the Lorenz equations [100]. All $\sigma, r, b > 0$, but usually $\sigma = 10$, $b = 8/3$ and r is varied. This system exhibits chaotic behavior for $r = 28$. The field

$$F_1 = -y - z, \qquad F_2 = x + ay, \qquad F_3 = b + cz - zx$$

defines the Rössler system [128]. Rössler studied the chaotic attractor with $a = 0.2$, $b = 0.2$, and $c = 5.7$. These Lorenz and Rössler systems defined by equations (2) and (4) are non-Hamiltonian and dissipative. The systems demonstrate a chaotic behavior for some values of parameters.

4. Non-Hamiltonian and dissipative classical systems

Let $A = A(\mathbf{x})$ be a smooth function on $\mathcal{M}$. Equation (1) gives

$$\frac{d}{dt}A = (\mathbf{F}, grad\, A), \tag{5}$$

where the brackets is a scalar product. We can define the operator $\mathcal{L} = (\mathbf{F}, \nabla_{\mathbf{x}})$, where $\nabla_{\mathbf{x}}$ is the nabla operator.

(1) For globally Hamiltonian systems, $\mathcal{L}$ is an inner derivation operator, i.e., there is $H \in \mathcal{M}$ such that

$$\mathcal{L} = \{H, \cdot\}, \tag{6}$$

where $\{\ ,\ \}$ is a Poisson bracket, and H is a unique single-valued function on $\mathcal{M}$.

(2) A locally Hamiltonian system is characterized by the conditions

$$Z_{\mathcal{L}}(A, B) = \mathcal{L}(AB) - (\mathcal{L}A)B - A(\mathcal{L}B) = 0, \tag{7}$$

$$J_{\mathcal{L}}(A, B) = \mathcal{L}\big(\{A, B\}\big) - \{\mathcal{L}A, B\} - \{A, \mathcal{L}B\} = 0 \tag{8}$$

for all real-valued smooth functions $A = A(\mathbf{x})$ and $B = B(\mathbf{x})$ on $\mathcal{M}$. Equations (7) and (8) can be used as a definition of locally Hamiltonian systems.

These equations mean that $\mathcal{L}$ is a derivation operator. In general, the derivation operator is not inner. For example, every derivation $\mathcal{L}$ of polynomial A in real variables q, p can be presented in the form

$$\mathcal{L}A = \{H, A\} + b\left(A - ap\frac{\partial A}{\partial p} - (1-a)q\frac{\partial A}{\partial q}\right),$$

where a, b are numbers. Thus every derivation of polynomial is a sum of an inner derivation $\{H, A\}$ and an explicitly determined outer derivation. (However this decomposition is not unique.) As a result, locally Hamiltonian system is not equivalent to globally Hamiltonian.

(3) For non-Hamiltonian systems, there exist functions $A(\mathbf{x})$ and $B(\mathbf{x})$ and points $\mathbf{x}$, such that equations (6) and (7) are not satisfied. We can use this property as a definition of classical non-Hamiltonian systems.