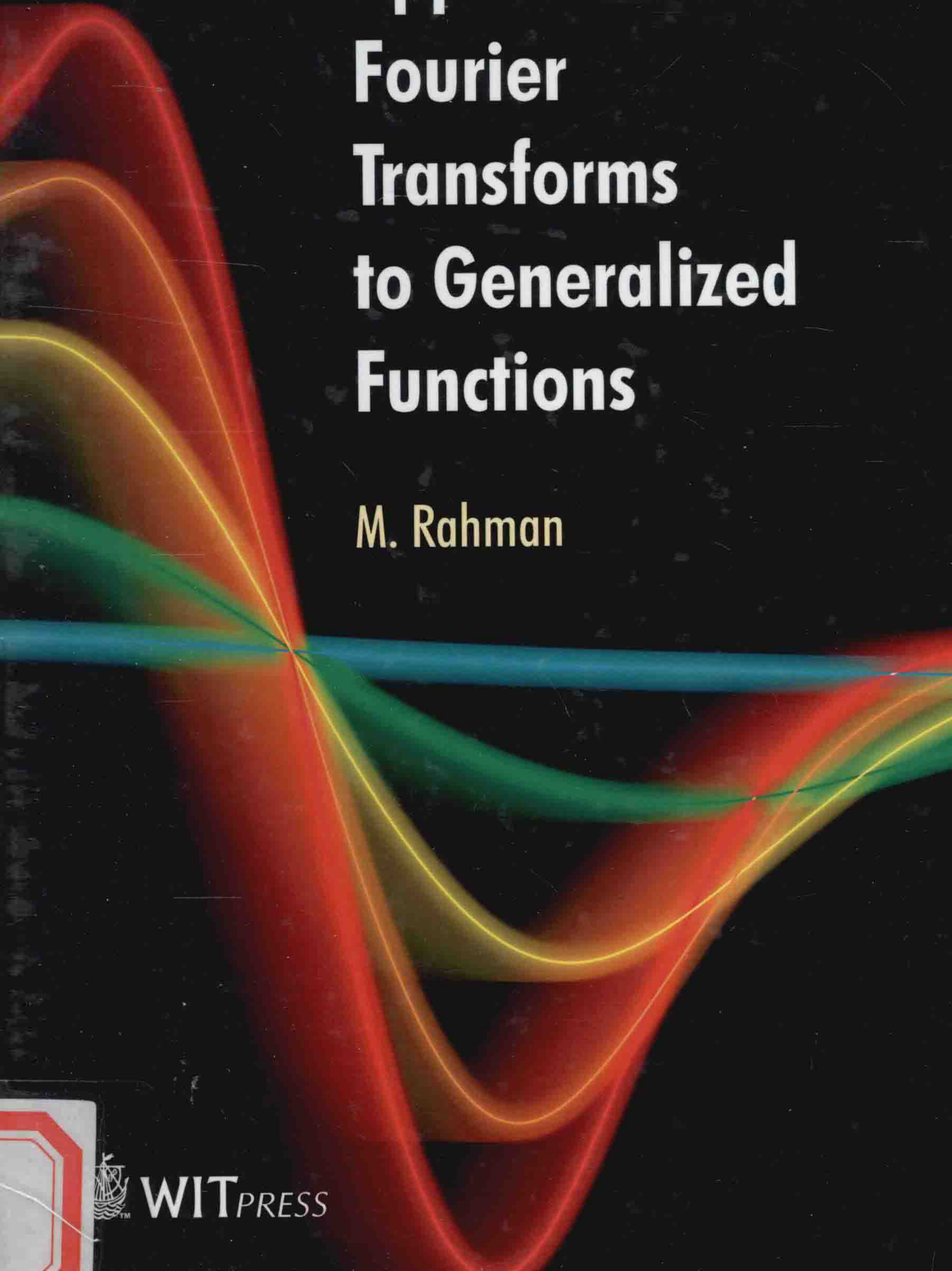
Applications of Fourier Transforms to Generalized Functions

M. Rahman



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Preface

The generalized function is one of the important branches of mathematics that has enormous application in practical fields. Especially, its applications to the theory of distribution and signal processing are very much noteworthy. The method of generating solutions is the Fourier transform, which has great applications to the generalized functions. These two branches of mathematics are very important for solving practical problems. While I was at Imperial College London (1966–1969), I attended many lectures delivered on fluid mechanics topics by Sir James Lighthill, FRS. At that time I was unable to understand many of the mathematical ideas in connection with the generalized function and why we need this abstract mathematics in the applied field. I tried to follow Lighthill's book *An Introduction to Fourier Analysis and Generalized Functions*, published by Cambridge University Press, 1964. His book is very compact (only 79 pages) and extremely stimulating, but he has written it so elegantly that unless one has good mathematical background, the book is very hard to follow. I understand that a non-expert reader will find the book very hard to follow because of its compactness and too many cross-references. Mathematical details are very minimal and he sequentially explains from one step to another skipping many intermediate steps by the cross references. Lighthill followed the ideas originally described by Professor George Temple's *Generalized Functions, Proc. Roy. Soc. A,* **228**, *175–190, (1955)*. Lighthill kept the theory part as described by Temple. In Dalhousie University I used to give a course on *Mathematical Methods and their Applications* to the undergraduate and graduate students for several years. I used Fourier transforms and generalized functions in that course. To make it understandable to the student I had to take recourse to some engineering textbooks where the applications are found in this subject. I followed some engineering application of generalized functions and its solution technique using the Fourier transform method.

This book grew partly out of my course given to the undergraduate and graduate students at Dalhousie University, Halifax, Nova Scotia, Canada; and partly from reading the books by Temple and Lighthill. This book explains clearly the intermediate steps not found in any other book. The book leans heavily towards Lighthill's book, but I have bridged the gap of mathematical deductions by clearly manifesting every important step with illustrations and mathematical tables. I think a layman can also follow my book without much difficulty. I must admit that this book is written in such a way as if I have revisited Lighthill's book *An Introduction to Fourier Analysis and Generalized Functions*. This book, hopefully, will be useful to the non-expert and also the experts alike. With this intention, the book is prepared in my own way collecting some additional material from some other textbooks including Professor D.S. Jones' book on *Generalized Functions*, published by McGraw-Hill Book Company, New York, 1966. I have borrowed some ideas from Professor Jones' book. Specially, I borrowed some important practical unsolved examples that I solved myself for the benefit of the reader. It is my hope that the reader will gain some insight about this important but esoteric mathematical subject.

The first chapter of the book deals with the introductory concept of Fourier series, Fourier integrals, Fourier transforms and the generalized function. The theoretical development of the Fourier transform is described and the first generalized function is defined with some illustrations. Some important examples are manifested in this chapter. Some interesting exercises are included at the end of the chapter.

Chapter 2 deals with the formal definition of the generalized function. A clear-cut definition of a good function and a fairly good function as illustrated by Lighthill is demonstrated in this chapter. The difference between an ordinary function and a generalized function is given with some examples. Even and odd generalized functions are clearly defined. The chapter ends with some useful exercises.

Chapter 3 consists of Fourier transforms of particular generalized functions. This chapter deals with the integral power of an algebraic function, non-integral powers, the Fourier transforms of $x^n \ell n|x|$, $x^{-m}\ell n\ |x|$, $x^{-m}\ell n|x|sgn(x)$ together with the summary of results of Fourier transforms. The chapter concludes with some exercises.

Asymptotic estimation of Fourier transforms are discussed in details in Chapter 4. First we have defined the Riemann-Lebesgue lemma which is important to obtain the asymptotic value of a generalized function. The asymptotic expression of the Fourier transform of a function with a finite number of singularities is discussed. We demonstrated solutions of some generalized functions using asymptotic expressions. Fourier transforms play a major role. Some important numerical solutions of some integrals are listed in Table 4.1. Whereas Table 4.2 contains a short list of Fourier transforms of 18 important generalized functions at a glance. The chapter ends with some important exercises.

Chapter 5 contains the Fourier series as a series of generalized functions. We demonstrated how to evaluate the coefficients of a trigonometric series. Some practical examples such as Poisson's summation formula and the asymptotic behaviour of the coefficients in a Fourier series are illustrated. This chapter concludes with some exercises.

We conclude the book (Chapter 6) with an important topic concerning the fast Fourier transform. It is a numerical procedure which is fast, accurate and efficient to determine the Fourier coefficients that are the Fourier transforms using an algorithm developed by Cooley and Tukey in 1965. Some preliminary studies of the Fourier transform with ample examples are also demonstrated in this chapter by using analytical and graphical methods. We have not reiterated the algorithm of Cooley and Tukey, rather we have given a numerical view of how it works, citing a practical example in the study of wave energy spectrum density as illustrated elegantly by Chakrabarti (1987). A handful of exercises are included and some references are cited at the end of the chapter.

The book concludes with three appendices. Appendix A deals with Fourier transforms of some important generalized functions. Appendix B is concerned with some important properties of Dirac delta $\delta(x)$ functions and Appendix C contains a comprehensive list of some important references concerning with the generalized functions and the application of the fast Fourier transform for further reading. A subject index is also included at the end of the book.

While it has been a joy to write such a comprehensive book for a long period of time, the fruits of this labour will hopefully be in learning of the enjoyment and benefits realized by the reader. Thus the author welcomes any suggestions for the improvement of the text.

Matiur Rahman, 2011
Halifax, Canada

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I am extremely grateful to Dr Michael Shepherd, Dean of Faculty of Computer Science at Dalhousie University for allowing me to use some facilities to complete this manuscript. Thanks are also extended to Dr Denis Riordan, Professor and Associate Dean of Faculty of Computer Science for his encouragement and constructive and favorable comments about the manuscript.

This book is primarily derived from Lighthill's book on *Introduction to Fourier analysis and generalised functions* published by Cambridge University Press in 1958 and subsequently reprinted in 1964. Thus, Cambridge University Press deserves my appreciations for the use of ideas and concepts which help me develop the present manuscript. WIT Press is gratefully acknowledged for its superb job in producing such a beautiful book.

Contents

1 Introduction

1.1 Preliminary remarks

Generalized functions are an important area of mathematical sciences. They have a wide range of applications. I taught at Dalhousie University several applied mathematics courses including mathematical methods which contain Laplace transforms, Fourier series and integrals, and Fourier transforms. These subject matters were developed during the courses at the undergraduate and graduate levels. The generalized function was used in connection with the problems of distribution theory and signal processing. It seems that a satisfactory account must make use of functions, such as the delta function of Dirac, which are outside the usual scope of functional theory. The *Theorie des distributions* by Schwartz (1950–1951) is one of the books that has developed a detailed theory of generalized functions. Professor Temple (1953, 1955) has given a version of the theory of generalized functions which appears to be more readily intelligible to senior undergraduates. Temple's book curtails the labour of understanding Fourier transforms. His book makes available a technique for the asymptotic estimation of Fourier transforms which seems superior to previous techniques. This is an approach in which the theory of Fourier series appears as a special case, the Fourier transform of a periodic function being a *row of delta functions*. Lighthill (1964) discussed this matter elaborately.

The main purpose of this book is to explain some of the very abstract material presented by Lighthill in his book. I think it will be a very useful addition to the literature and will help the graduate students of our future generation to have a clear-cut exposition of the subject of generalized functions and Fourier transforms. This book not only covers the principal results concerning Fourier transforms and Fourier series, but also serves as an introduction to the theory of generalized functions that are used in Fourier analysis. It contains simple properties without rigorous mathematical proofs.

1.2 Introductory remarks on Fourier series

A Fourier series is a representation of a periodic function $f(x)$ of period 2ℓ, say, which means that $f(x) = f(x+2\ell) = \cdots = f(x+2n\ell)$, for $n = 1, 2, 3, \ldots$, as a linear combination of all *cosine* and *sine* functions which have the same period, say, as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(n\pi x/\ell)\,dx, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(n\pi x/\ell)\,dx, \quad n = 1, 2, \ldots. \tag{1.1}$$

The fundamental period in the trigonometric expression on the right-hand side is 2ℓ, because for $n=1$, $\cos\left(\frac{\pi x}{\ell}\right) = \cos\left(\frac{\pi x}{\ell} + 2\pi\right) = \cos\left(\frac{\pi}{\ell}(x+2\ell)\right)$, and similarly for $\sin\left(\frac{\pi x}{\ell}\right)$. The constant a_0 has no fundamental period but it is a periodic function with any natural number including 2ℓ. Fourier series in this sense are used for analysing oscillations periodic in time, or waveforms periodic in space, and also for representing functions of plane or cylindrical coordinates, when x in eqn (1.1) becomes the polar angle θ, and the period 2ℓ becomes 2π. It is interesting to note here that

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \tag{1.2}$$

such that $f(\theta) = f(\theta + 2\pi)$.

The series (1.1) can be written, more compactly, in the complex form as follows:

$$\begin{aligned}
f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{e^{in\pi x/\ell} + e^{-in\pi x/\ell}}{2} + b_n \frac{e^{in\pi x/\ell} - e^{-in\pi x/\ell}}{2i} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-in\pi x/\ell} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} + \sum_{n=-\infty}^{-1} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} \\
&= \frac{a_0}{2} + \sum_{n=-\infty}^{\infty} \frac{a_n - ib_n}{2} e^{in\pi x/\ell} \\
&= C_0 + \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell} \\
&= \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell},
\end{aligned}$$

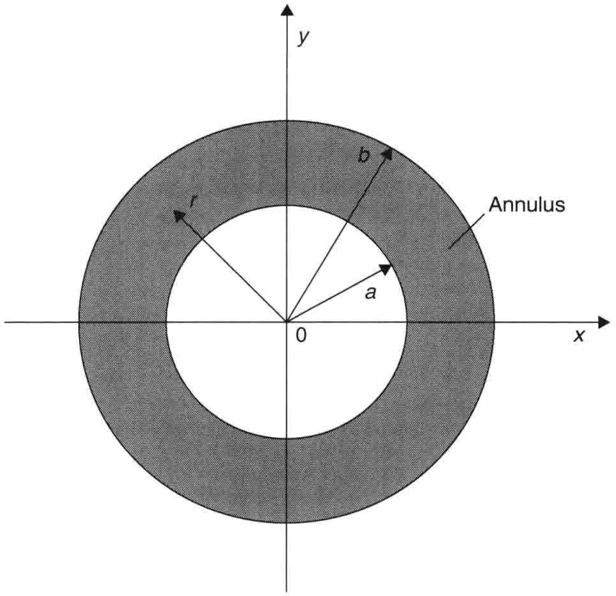

Figure 1.1: An annular region between two concentric circles of radii $r = a$ and b $(a \leq r \leq b)$.

where $C_0 = \frac{a_0}{2}$, $C_n = (a_n - ib_n)/2$ and $C_{-n} = (a_n + ib_n)/2 = C_n^*$, in which C_n^* is a complex conjugate. It can be easily verified that $C_n + C_n^* = a_n$ and $C_n - C_n^* = -ib_n$. With this information the trigonometric series (periodic function) can be represented as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/\ell},$$

$$C_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell}\, dx. \tag{1.3}$$

One great advantage of expressing a function in terms of cosine and sine functions, or even more in terms of exponentials, is the simple behaviour of these functions under the various operations of analysis, notably differentiations and integrations. The Fourier series expression (1.3) may be used for some practical problems. For example, solutions to Laplace's equation in plane polar coordinates r; θ, which are periodic in θ with period 2π, thus, representing solutions which are one-valued in an annulus with the centre as the origin, may be written as a boundary value problem as illustrated in Figure 1.1.

The partial differential equation is

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0, \quad 0 \leq \theta \leq 2\pi, \ \ a \leq r \leq b. \tag{1.4}$$

The boundary conditions are

$$r = a: \quad f(a,\theta) = g(\theta), \tag{1.5}$$

$$r = b: \quad \frac{\partial f}{\partial r}(b,\theta) = h(\theta). \tag{1.6}$$

The periodic solution to eqn (1.4) may be written as

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} C_n(r)e^{in\theta}. \tag{1.7}$$

If we substitute this in eqn (1.4), assuming that we can differentiate term by term, we obtain

$$\sum_{n=-\infty}^{\infty} \left[\frac{d^2C_n}{dr^2} + \frac{1}{r}\frac{dC_n}{dr} - \frac{n^2}{r^2}C_n\right]e^{in\theta} = 0. \tag{1.8}$$

Now if we assume that expressions of functions by such trigonometric series are unique, then a series which vanishes identically must have vanishing coefficients, while an ordinary differential equation is obtained for C_n, $\frac{d^2C_n}{dr^2} + \frac{1}{r}\frac{dC_n}{dr} - \frac{n^2}{r^2}C_n = 0$, whose general solution is

$$C_n(r) = A_n r^n + B_n r^{-n}. \tag{1.9}$$

Therefore, we have

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} [A_n r^n + B_n r^{-n}]e^{in\theta}. \tag{1.10}$$

If the boundary conditions (1.5) and (1.6) are applied on the circles $r = a$ and b, respectively, to determine the unknown constants A_n and B_n, we must express $g(\theta)$ and $h(\theta)$ in Fourier series so that

$$g(\theta) = \sum_{n=-\infty}^{\infty} g_n e^{in\theta}, \quad h(\theta) = \sum_{n=-\infty}^{\infty} h_n e^{in\theta}. \tag{1.11}$$

Then we have two algebraic equations

$$\begin{aligned} A_n a^n + B_n a^{-n} &= g_n, \\ A_n b^n - B_n b^{-n-1} &= \frac{h_n}{n}, \end{aligned} \tag{1.12}$$

the solutions to which are given by

$$\begin{aligned} A_n &= [b^{-n-1}g_n + a^{-n-1}(h_n/n)]/D, \\ B_n &= [b^{n-1}g_n - a^n(h_n/n)]/D, \end{aligned} \tag{1.13}$$

where $D = a^n b^{-n-1} + a^{-n-1}b^{n-1}$.

This example is so simple that it could be treated in many different ways, but clearly, the procedure is directly applicable in most complicated problems, provided always that the argument of the Fourier series (here θ) varies independently of the other variables from the boundary conditions. This example makes it clear that a satisfactory Fourier series theory will be one in which term-by-term differentiation and unique determination of coefficients for a given function are both possible. These two requirements had never been simultaneously satisfied by any of the Fourier series theories until the *generalized function* approach was developed.

1.3 Half-range Fourier series

Some concepts about the half-range Fourier series and quarter-range Fourier series are described by the following example. If a partial differential equation is to be solved in a region part of whose boundary consists of the lines (or planes) $x = 0$ and ℓ, then the argument is usually presented as follows. The only cosines or sines which satisfy the boundary condition $f = 0$ both at $x = 0$ and ℓ are $\sin(n\pi x/\ell)$ $(n = 1, 2, 3, \ldots)$, so that the half-range Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/\ell) \quad (0 < x < \ell) \tag{1.14}$$

is applicable when these are the boundary conditions. Alternatively, if $\frac{\partial f}{\partial x} = 0$ at $x = 0$ and ℓ, then the half-range Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/\ell) \quad (0 < x < \ell) \tag{1.15}$$

is applicable. Note that in the complex form (1.3), in these cases C_n is purely imaginary and real, respectively.

Again, if $f = 0$ at $x = 0$ and $\frac{\partial f}{\partial x} = 0$ at $x = \ell$, the quarter-range Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2\ell} \quad (0 < x < \ell), \tag{1.16}$$

containing an even selection of the terms in a Fourier series of larger period 4ℓ, is applicable. These predictions can be easily verified by considering the problem of oscillations of a one-dimensional string of length ℓ fixed at both ends by using appropriate boundary conditions.

1.3.1 Verification of conjecture (1.14)

To verify the solution (1.14), we consider the following mathematical model. The partial differential equation with its boundary conditions is

$$\begin{aligned}\frac{\partial^2 f}{\partial t^2} &= c^2\frac{\partial^2 f}{\partial x^2} \quad (0 < x < \ell)\ (t > 0),\\ x=0:\quad f &= 0,\\ x=\ell:\quad f &= 0.\end{aligned} \tag{1.17}$$

The oscillatory solution in t is given by $f \sim f(x)e^{i\lambda t}$ such that the partial differential equation becomes an ordinary differential equation

$$\frac{d^2 f}{dx^2} + \mu^2 f = 0, \tag{1.18}$$

where $\mu^2 = \frac{\lambda^2}{c^2}$. Using the two boundary conditions we obtain the series solution as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/\ell) \quad (0 < x < \ell).$$

This is a half-range Fourier sine series and b_n is the Fourier coefficients as given in eqn (1.14). If the function is prescribed then b_n can be determined by using the orthogonal properties of the set $\{\sin n\pi x/\ell\}$, for $n = 1, 2, 3, \ldots$, defined in the interval $0 \le x \le \ell$. The formula for b_n is given by $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(n\pi x/\ell)\,dx$, $n = 1, 2, 3, \ldots$.

1.3.2 Verification of conjecture (1.15)

To verify the solution (1.15), we consider the following mathematical model. The partial differential equation with its boundary conditions is

$$\begin{aligned}\frac{\partial^2 f}{\partial t^2} &= c^2\frac{\partial^2 f}{\partial x^2} \quad (0 < x < \ell)\ (t > 0),\\ x=0:\quad \frac{\partial f}{\partial x} &= 0,\\ x=\ell:\quad \frac{\partial f}{\partial x} &= 0.\end{aligned} \tag{1.19}$$

The oscillatory solution in t is given by $f \sim f(x)e^{i\lambda t}$ such that the partial differential equation becomes an ordinary differential equation

$$\frac{d^2 f}{dx^2} + \mu^2 f = 0, \tag{1.20}$$

where $\mu^2 = \frac{\lambda^2}{c^2}$. Using the two boundary conditions we obtain the series solution as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/\ell) \quad (0 < x < \ell).$$

This is a half-range Fourier cosine series and a_n is the Fourier coefficients as given in eqn (1.15). If the function is prescribed then b_n can be determined by using the orthogonal properties of the set $\{\cos(n\pi x/\ell)\}$, for $n = 0, 1, 2, 3, \ldots$, defined in the interval $0 \le x \le \ell$. The formula for a_n is given by $a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos(n\pi x/\ell)\,dx, n = 0, 1, 2, 3, \ldots$.

1.3.3 Verification of conjecture (1.16)

To verify the solution (1.16), we consider the following mathematical model. The partial differential equation with its boundary conditions is

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} &= c^2 \frac{\partial^2 f}{\partial x^2} \quad (0 < x < \ell) \ (t > 0), \\ x = 0: \quad f &= 0, \\ x = \ell: \quad \frac{\partial f}{\partial x} &= 0. \end{aligned} \tag{1.21}$$

The oscillatory solution in t is given by $f \sim f(x)e^{i\lambda t}$ such that the partial differential equation becomes an ordinary differential equation

$$\frac{d^2 f}{dx^2} + \mu^2 f = 0, \tag{1.22}$$

where $\mu^2 = \frac{\lambda^2}{c^2}$. Using the two boundary conditions we obtain the series solution as

$$f(x) = \sum_{n=1}^{\infty} b_n (\sin(2n-1)\pi x/2\ell) \quad (0 < x < \ell).$$

This is a quarter-range Fourier series and b_n is the Fourier coefficients as given in eqn (1.16). If the function is prescribed then b_n can be determined by using the orthogonal properties of the set $\{\sin((2n-1)\pi x/2\ell)\}$, for $n = 1, 2, 3, \ldots$, defined in the interval $0 \le x \le \ell$. The formula for b_n is given by $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin((2n-1)\pi x/2\ell)\,dx,\ n = 1, 2, 3, \ldots$.

Remark

These series can be approached in a slightly more useful manner as described below. To satisfy the boundary conditions $f(0) = 0$ and $f(\ell) = 0$, an *odd periodic function* $f(x)$ of period 2ℓ is introduced. Its Fourier series (1.1) then contains only odd terms and reduces to eqn (1.14).