Eberhard Zeidler

Nonlinear Functional Analysis and its Applications III

Variational Methods and Optimization

非线性泛函分析及其应用

第3卷

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Nonlinear Functional Analysis and its Applications

III: Variational Methods and Optimization

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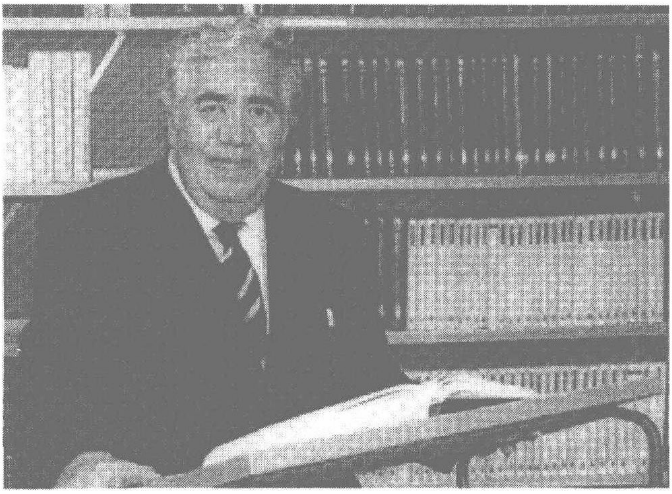

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影印版前言

自 1932 年，波兰数学家 Banach 发表第一部泛函分析专著"Théorie des operations linéaires"以来，这一学科取得了巨大的发展，它在其他领域的应用也是相当成功。如今，数学的很多领域没有了泛函分析恐怕寸步难行，不仅仅在数学方面，在理论物理方面的作用也具有同样的意义，M. Reed 和 B. Simon 的"Methods of Modern Mathematical Physics"在前言中指出："自 1926 年以来，物理学的前沿已与日俱增集中于量子力学，以及奠定于量子理论的分支：原子物理、核物理固体物理、基本粒子物理等，而这些分支的中心数学框架就是泛函分析。"所以，讲述泛函分析的文献已浩如烟海。而每个时代，都有这个领域的代表性作品。例如上世纪 50 年代，F. Riesz 和 Sz.-Nagy 的《泛函分析讲义》(中译版，科学出版社，1985)，就是那个时代的一部具有代表性的著作；而 60 年代，N. Dunford 和 J. Schwartz 的三大卷"Linear Operators"则是泛函分析学发展到那个时代的主要成果和应用的一个较全面的总结。泛函分析一经产生，它的发展就受到量子力学的强有力的推动，上世纪 70 年代，M. Reed 和 B. Simon 的 4 卷"Methods of Modern Mathematical Physics"是泛函分析对于量子力学应用的一个很好的总结。

呈现在我们眼前这部 5 大卷鸿篇巨制——E. Zeidler 的"Nonlinear Functional Analysis and its Applications"是非线性泛函分析到了上世纪 80 年代的主要成果和最典型应用的一个全面的论述，是一部百科全书。该书写作思想是：

(1) 讲述什么样的概念是基本的具有支配地位的概念，它们之间的关系是什么?

(2) 上述思想与经典分析以及线性泛函分析已有结果的关系是什么?

（3）　最典型的应用是什么？

一般的泛函分析书往往注重抽象理论的阐述，写应用常常不够详尽。而 Zeidler 这部书大为不同，其最大的特点是，书中讲了大量的各方面的应用，而且讲得非常清楚深入。

首先，这部书讲清楚了泛函分析理论对数学其他领域的应用。例如，第 2A 卷讲述线性单调算子。他从椭圆型方程的边值问题出发，讲问题的古典解，由于具体物理背景的需要，问题须作进一步推广，而需要讨论问题的广义解。这种方法背后的分析原理是什么？其实就是完备化思想的一个应用！将古典问题所依赖的连续函数空间，完备化成为 Sobolev 空间，则可讨论问题的广义解。在这种讨论中间，我们可以看到 Hilbert 空间的作用。书中不仅有这种理论讨论，而且还讲了怎样计算问题的近似解（Ritz 方法）。

其次，这部书讲清楚了分析理论在诸多领域（如物理学、化学、生物学、工程技术和经济学等等）的广泛应用。例如，第 3 卷讲解变分方法和优化，它从函数极值问题开始，讲到变分问题及其对于 Euler 微分方程和 Hammerstein 积分方程的应用；讲到优化理论及其对于控制问题（如庞特里亚金极大值原理）、统计优化、博弈论、参数识别、逼近论的应用；讲了凸优化理论及应用；讲了极值的各种近似计算方法。比如第 4 卷，讲物理应用，写作原理是：由物理事实到数学模型；由数学模型到数学结果；再由数学结果到数学结果的物理解释；最后再回到物理事实。

再次，该书由浅入深地讲透了基本理论的发展历程及走向，它既讲清楚了所涉及学科的具体问题，也讲清楚了其背后的数学原理及其作用。数学理论讲得也非常深入，例如，不动点理论，就从 Banach 不动点定理讲到 Schauder 不动点定理，以及 Bourbaki-Kneser 不动点定理等等。

这套书的写作起点很低，具备本科数学水平就可以读；应用都是从最简单情形入手，应用领域的读者也可以读；全书材料自足，各部分又尽可能保持独立；书后附有极其丰富的参考文献及一些文献评述；该书文字优美，引用了许多大师的格言，读之你会深受启发。这套书的优点不胜枚举，每个与数理学科相关的人，搞理论的，搞应用的，搞研究的，搞教学的，都可读该书，哪怕只是翻一翻，都不会空手而返！

全书共有 4 卷（5 本）：

第 1 卷　不动点定理　　　第 2 卷　A. 线性单调算子，B. 非线性单调算子

第 3 卷　变分方法与优化　　第 4 卷　数学物理的应用

Zeidler 教授著述很多，他后来于 90 年代又写了两本“Applied Functional Analysis”（Springer-Verlag, Applied Mathematical Sciences, 108,109），篇幅虽然比眼前这套书小了很多，但特点没有变。近期，他又在写 6 大卷“Quantum Field Theory”，第 1 卷“Basics in Mathematics and Physics, a Bridge between Mathematicians and Physicists”，已经由 Springer-Verlag 出版社出版。

非常感谢刘景麟对本文建议。

南京理工大学　黄振友

Dedicated in gratitude to my teacher

Professor Herbert Beckert

Preface

> As long as a branch of knowledge offers an abundance of problems, it is full of vitality.
>
> David Hilbert

Over the last 15 years I have given lectures on a variety of problems in nonlinear functional analysis and its applications. In doing this, I have recommended to my students a number of excellent monographs devoted to specialized topics, but there was no complete survey-type exposition of nonlinear functional analysis making available a quick survey to the wide range of readers including mathematicians, natural scientists, and engineers who have only an elementary knowledge of linear functional analysis. I have tried to close this gap with my five-part lecture notes, the first three parts of which have been published in the Teubner-Texte series by Teubner-Verlag, Leipzig, 1976, 1977, and 1978. The present English edition was translated from a completely rewritten manuscript which is significantly longer than the original version in the Teubner-Texte series. The material is organized in the following way:

Part I: Fixed Point Theorems.
Part II: Monotone Operators.
Part III: Variational Methods and Optimization.
Parts IV/V: Applications to Mathematical Physics.

The exposition is guided by the following considerations:

(α) What are the supporting basic ideas and what intrinsic interrelations exist between them?
(β) In what relation do the basic ideas stand to the known propositions of classical analysis and linear functional analysis?
(γ) What typical applications are there?

Special emphasis is placed on motivation. The reader should always have the feeling that the theory is not developed for its own sake but rather for the effective solution of concrete problems. At the same time I try to outline a variegated picture of the subject matter which ranges from the fundamental questions of set theory (the Bourbaki–Kneser fixed point theorem) to concrete numerical methods, encompassing numerous applications to physics, chemistry, biology, and economics. The reader should see mathematics as a unified whole, with no separation between pure and applied mathematics. At the same time we show how deep mathematical tools can be used in the natural sciences, engineering, and economics. The development of nonlinear functional analysis has been influenced in an essential way by complicated natural scientific questions; the close contact with the natural sciences and other sciences will also be of great significance for the development of nonlinear functional analysis. In our exposition, the use of analytic tools stands in the foreground, but we also seek to show connections with algebraic and differential topology. For instance, Sections 37.27 and 37.28 contain an introduction to Morse theory as well as to singularity and catastrophe theory. To reach the largest possible readership and to fashion a self-contained exposition, important tools from linear functional analysis are provided in the appendices to Parts I and II. These are presented so that readers with a skimpy background can familiarize themselves with this material. We forego, at the outset, the greatest possible generality, but rather seek to expose the simple intrinsic nucleus without trivializing it. According to the author's experience, it is easier for the student to generalize familiar mathematical ideas to a more general situation than to elicit the basic idea from a theorem that is formulated very generally and burdened with many technical details. The teacher must help him in that task. In order to make it easier for the reader to grasp the central results, a number of propositions have been listed in a separate section called *List of Theorems* to be found on page 643. It is clear that this procedure is not entirely free of arbitrariness. However, we hope that the lists of Theorems for Parts I–V provide an overview of the essential substance of nonlinear functional analysis. Furthermore, since, in the experience of the author, it is frequently difficult, because of a flood of details, for the student to recognize the interrelationships between different questions and the general strategies for the solution of problems, special emphasis is placed on these interrelationships.

We have given a general overview of the content of Parts I–V and the basic idea of nonlinear functional analysis in the Preface and in the introduction to Part I. The present Part III consists of the following topics:

(α) Introduction to the subject.
(β) Two fundamental existence and uniqueness principles.
(γ) Extremal problems without side conditions.
(δ) Extremal problems with smooth side conditions.
(ε) Extremal problems with general side conditions.

(ζ) Saddle points and duality.
(η) Variational inequalities.

In the introduction, and in the schematic survey in Fig. 37.1 on page 3, we give an overview of the interrelationships between various extremal problems. In the comprehensive introductory Chapter 37, we present many simple, but typical, examples that are representative of those concrete problems that have played a central role in the historical development of the subject. In order to obtain an impression of the extraordinary variety of problems involved, the reader should glance at the list of subjects for Chapter 37 that appears in the Contents. In the immediately following chapters it is our chief concern to show the reader that these problems can be handled with the aid of a *unified theory* of extremal problems. The essence of this unified theory consists of a small number of fundamental principles of functional analysis. The title of Part III, Variational Methods and Optimization, indicates that we consider aspects of the classical calculus of variations as well as modern optimization theory and their interrelationships. By working out the supporting ideas and general fundamental principles, we also wish to help the reader obtain an understanding of the substance of the extraordinarily comprehensive and turbulently accumulating literature on extremal problems, to classify these works according to their ideas, and to note the emergence of new ideas.

Each of the 21 chapters is self-contained. Each begins with motivations, heuristic considerations, and indications of the typical problems to be investigated and contains the most important theorems and definitions together with elucidating examples, figures, and typical applications. We also do not shun citing very simple examples in the interest of the reader. Furthermore, we always try to penetrate as quickly as possible to the heart of the matter. We try to achieve the situation where the reader knows at each phase of the book what concrete applications the general considerations allow. In general, a very careful selection of the material had to be made because one could write each chapter as a special monograph and, to some extent, such monographs already exist. Here, we describe the applications to nonlinear differential and integral equations, differential inequalities, one-dimensional and multidimensional variational problems, linear and convex optimization problems, problems in approximation theory and game theory, continuous and discrete control problems for ordinary and partial differential equations, and also consider important approximation methods. In particular, in Section 37.29, we explain the basic ideas of 10 important methods and principles for the construction of approximation methods. In the introduction to Part I we have already pointed out that in numerical methods the devil rides high on detail. However, general principles and theoretical investigations of approximation methods within the setting of numerical functional analysis are useful for recognizing the basic ideas and for arranging the abundance of concrete numerical methods into a unified point of view. We examine a number of more profound applications of nonlinear functional analysis to mathematical physics in Parts IV and V.

At the end of each chapter the reader will find problems and references to the literature. The problems vary considerably in their degree of difficulty:

(α) Problems without asterisks serve as drills in the material presented and require no additional tools.
(β) Problems with asterisks are more difficult—additional ideas are required to solve them.
(γ) Problems with double asterisks are very difficult—one needs substantial additional information to solve them.

Each problem contains either a solution or a precise reference to the monograph or original work in which the solution can be found. Moreover, we try to clarify the meaning of the results with explanatory remarks. The problems with one or two asterisks are in part so devised that they present targeted references to the literature on important extensions of results or they serve to extend the reader's mathematical horizon. A number of topics will be treated supplementarily in the problem collections. These topics are particularly extensive in Chapter 40, where we try to sketch for the reader a line of development from the classical calculus of variations and from geometrical optics up to the modern theory of Fourier integral operators. In this we let ourselves be led by the experience that the penetration of a complicated theory is made easier for the student when she/he has an ultimate goal from the beginning and knows the connection between the goal and the simpler questions familiar to her/him.

The references to the literature at the end of each chapter are styled as follows: Krasnoselskii (1956, M, B, H), etc. The year refers to the list of literature at the end of the book. Furthermore, the capital Latin letters mean:

M: monograph;
L: lecture notes;
S: survey;
P: proceedings;
B: the cited work contains a comprehensive bibliography;
H: the cited work contains references to the historical development of the subject.

In this connection, the references to the literature are at the same time supplied with clarifying captions which explain the interrelationship between the works cited. On page 166 one finds "Recent trends". From the abundance of available literature we have made a careful but necessarily subjectively biased selection, which in the author's opinion will easily afford the reader as comprehensive a picture as possible concerning the farther-leading results. In this, the emphasis lies naturally on the surveys and monographs. However, we also cite a number of classical works which were of special significance for the development of the subject. We recommend that the reader glance at several of these works in order to obtain an active impression of the genesis of new results and of the historical development of mathematics. Unfortunately, in order to keep the list of literature within tolerable bounds, we had to forego listing many important references.

In the choice of the presentation it was taken into consideration that in general no book is read completely from beginning to end. We hope that even a quick skimming of the text will suffice for one to grasp the essential contents. To this end, we recommend reading the introductions to the individual chapters, the definitions, the theorems (without proofs), and the examples (without proofs) as well as the comments in the text between these definitions, theorems, etc., which point out the meaning of the individual results. The reader who does not have time to solve the problems should, however, briefly scrutinize the captions to the problems and the adjoining remarks, which elucidate the meaning of the formulation of the problems and the interrelationships. The reader who is interested in supplementary problem material can try to prove independently all of the examples in the text without referring to the given proof. Moreover, in the references to the literature in Section 37.29, books are cited in which the reader will find comprehensive collections of exercises that as a rule are not too difficult. All hypotheses both in the theorems and in the examples are explicitly stated so that the reader avoids a time-consuming search for the assumptions in the antecedent text. We have taken pains to reduce the number of definitions to a minimum in order not to burden the reader with too many concepts. On page xii one finds a list of the most important definitions. In order to clarify interrelationships, several assertions that belong together are at times combined into a single theorem. In this form of exposition, we have also kept in mind the natural scientist and the engineer who want primarily to gain information on which mathematical tools are available for the various nonlinear problems. We recommend Chapter 37 to the reader who wishes to examine the class of problems which the general theory allows one to treat. However, it suffices to glance at this comprehensive chapter, because references will later be made at the appropriate places. The reader whose priority is to become acquainted with the theoretical framework can immediately begin with Chapter 38 and, on first reading, omit the sections in the individual chapters that are devoted to applications.

Grasping the individual steps in the proofs as well as the essential ideas of the proofs is made easier by the careful organization of the proofs. It is a truism that only by a precise study of the proofs one can penetrate more deeply into a mathematical theory.

Part III is to a large extent independent of the other parts. However, where necessary, we do refer to particular results of the other parts. Note that several auxiliary tools are made available in Parts I and II (basic information concerning linear functional analysis, Sobolev spaces, etc.). We formulate a number of results for locally convex spaces. The reader who is not familiar with this material can orient himself by reading the appendix to Part I or replace the concept of a locally convex space by that of a Banach or Hilbert space. Dual pairs are important for duality theory. We explain this concept in the appendix to Part III. The reference A_i (20) relates to (20) in the appendix to the ith part. (37.20) is formula (20) in Chapter 37. Within a particular chapter, we forego giving the chapter number of the equation. In each chapter, theorems are distinguished by capital letters, so that, for instance, "Theorem 57.B in Section 57.5" means the second theorem in Chapter 57, located in Section 5 of that chapter. Propositions, lemmas, corollaries, definitions, remarks, conventions, counterexamples, standard examples, and examples are numbered consecutively in each chapter—for example, in Chapter 41 one finds Definition 41.1, Proposition 41.2, Corollary 41.3, etc., in that order. The end of a proof is indicated by the symbol □. We subdivide the chapters among the five separate parts of this work in the following way:

Part I: Chapters 1–17.
Part II: Chapters 18–36.
Part III: Chapters 37–57.
Part IV: Chapters 58–79.
Part V: Chapters 80–100.

A list of symbols used can be found on page 637. We have taken pains to employ the notation that is generally used. To avoid confusion, we point out several peculiarities at the beginning of the list of symbols on page 637. A detailed subject index can be found on page 651. As far as abbreviations are concerned, we use only B-space (respectively, H-space) for Banach space (respectively, Hilbert space), F-derivative (respectively, G-derivative) for Fréchet derivative (respectively Gâteaux derivative) as well as M–S sequence for Moore–Smith sequence and L–S deformation for Ljusternik–Schnirelman deformation.

I have taken pains to write as interesting and diverse a book as possible. Of course, whether or not I have succeeded in this only the reader can decide.

I am indebted to numerous colleagues for interesting conversations and letters as well as for sending me articles and books—I thank them all heartily. I am especially grateful to my mentor Professor Herbert Beckert for all that I learned from him as a scientist and as a human being. I should like to dedicate the present volume to him. I cordially thank Paul H. Rabinowitz and the Department of Mathematics of the University of Wisconsin, Madison, for the invitation as guest resident scholar during the fall semester 1978. The very stimulating atmosphere in Madison influenced the final form of the exposition in an essential way. In the tasks of typing the manuscript and of making copies, I was supported in an amiable way by a number of colleagues, both male and female. I should like to very heartily thank Ursula Abraham, Sonja Bruchholz, Elvira Krakowitzki, Heidi Kühn, Hiltraud Lehmann, Karin Quasthoff, Werner Berndt, and Rainer Schumann. I would especially like to thank Rainer Schumann for a critical perusal of parts of the manuscript. The understanding and extensive support shown to

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Eberhard Zeidler
Leipzig
Spring 1984

Contents

TWO FUNDAMENTAL EXISTENCE AND UNIQUENESS PRINCIPLES