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Abstract

In this paper we extend the notion of a continuous bundle random dynamical system to the setting where the action of $\mathbb{R}$ or $\mathbb{N}$ is replaced by the action of an infinite countable discrete amenable group.

Given such a system, and a monotone sub-additive invariant family of random continuous functions, we introduce the concept of local fiber topological pressure and establish an associated variational principle, relating it to measure-theoretic entropy. We also discuss some variants of this variational principle.

We introduce both topological and measure-theoretic entropy tuples for continuous bundle random dynamical systems, and apply our variational principles to obtain a relationship between these of entropy tuples. Finally, we give applications of these results to general topological dynamical systems, recovering and extending many recent results in local entropy theory.

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Contents

CHAPTER 1

Introduction

In the early 1990s Blanchard introduced the concept of entropy pairs to search for satisfactory topological analogues of Kolmogorov systems [**3**, **4**]. Stimulated by these two papers, local entropy theory for continuous actions of a countable amenable group on compact metric spaces developed rapidly during the last two decades, see [**5**–**7**, **21**, **29**, **32**, **34**–**38**, **64**, **72**]. It has been studied for countable sofic group actions in [**76**] by the second author. For more details of the area, see for example Glasner's book chapter [**30**, Chapter 19] or the nice survey [**33**] by Glasner and Ye. Observe that, as shown by [**30**, Chapter 19] and [**33**] (and references therein), a detailed analysis of the local properties of entropy provides additional insight into the related global properties, and local properties of entropy can help us to draw conclusions for global properties.

The foundations of the theory of amenable group actions were set up in the pioneering paper [**62**] by Ornstein and Weiss, and further developed by Rudolph and Weiss [**65**] and Danilenko [**17**]. See also Benjy Weiss' lovely survey article [**71**]. Global entropy theory for amenable group actions has also been discussed by Moulin Ollagnier [**59**]. Other related aspects were discussed in [**18**, **20**, **31**, **60**, **61**, **66**, **69**]. The connection between local entropy and combinatorial independence across orbits of sets in dynamical systems was studied systematically by Kerr and Li in [**41**, **42**] for amenable group actions and in [**43**] for sofic group actions, and has been discussed by Chung and Li in [**14**] for amenable group actions on compact groups by automorphisms.

Our principal aim in this article is to extend the local theory of entropy to the setting of random dynamical systems of countable amenable group actions. To date, most discussions of random dynamical systems have concerned $\mathbb{R}$-actions, $\mathbb{Z}$-actions or $\mathbb{Z}_+$-actions. Furthermore, to the best of our knowledge, there has been little discussion of the local theory. In slightly more precise terms, we aim to make a systematic study of the local entropy theory of a continuous bundle random dynamical system over an infinite countable discrete amenable group.

In the setting of random dynamical systems, rather than considering iterations of just one map, we study the successive application of different transformations chosen at random. The basic framework was established by Ulam and von Neumann [**67**] and later by Kakutani [**40**] in proofs of the random ergodic theorem. Since the 1980s, mainly because of stochastic flows arising as solutions of stochastic differential equations, interest in the ergodic theory of random transformations has grown [**2**, **8**–**10**, **16**, **44**–**48**, **52**, **55**–**57**, **77**]. It was shown in [**8**] that the cornerstone for the entropy theory of random transformations is the Abramov-Rokhlin mixed entropy of the fiber of a skew-product transformation (cf [**1**]). Our main result, Theorem 7.1 establishes a variational principle for local topological pressure in this setting.

In the local entropy theory of dynamical systems as studied in [**30**, Chapter 19], [**33**] (and references therein) and [**38**], most significant results involving entropy pairs have been obtained using measure-theoretic techniques and a local variational principle initiated by [**5**].

Let G be an infinite countable discrete amenable group acting on a compact metric space X. Let $\mathcal{V}$ be a finite open cover of the space X, and ν a G-invariant Borel probability measure on X. Denote by $h_{\text{top}}(G, \mathcal{V})$ and $h_\nu(G, \mathcal{V})$ the topological entropy and measure-theoretic ν-entropy of $\mathcal{V}$, respectively. In [**38**] Huang, Ye and the second author of the paper proved the following version of local variational principle [**38**, Theorem 5.1]:

$$h_{\text{top}}(G, \mathcal{V}) = \max_{\nu \in \mathcal{P}(X,G)} h_\nu(G, \mathcal{V}), \tag{1.1}$$

where $\mathcal{P}(X, G)$ denotes the set of all G-invariant Borel probability measures ν on X. Subsequently, (1.1) was generalized by Liang and Yan [**53**, Corollary 1.2], recovering the global variational principle [**59**, Variational Principle 5.2.7] by Moulin Ollagnier. They showed that for each real-valued continuous function f over X,

$$P(f, \mathcal{V}) = \max_{\nu \in \mathcal{P}(X,G)} [h_\nu(G, \mathcal{V}) + \int_X f(x) d\nu(x)], \tag{1.2}$$

where $P(f, \mathcal{V})$ denotes the topological $\mathcal{V}$-pressure of f. We recover $h_{\text{top}}(G, \mathcal{V})$ when f is the constant zero function.

Remark that, in the local theory of entropy of dynamical systems, many variants of (1.1) and (1.2) have been discussed by [**5**, **12**, **32**, **35**, **36**, **39**, **64**, **75**], either for a $\mathbb{Z}$-action on compact metric spaces or for a factor map between topological $\mathbb{Z}$-actions.

Let the family $\mathbf{F}$, associated with $\mathcal{E} \in \mathcal{F} \times \mathcal{B}_X$, be a continuous bundle random dynamical system over a measure-preserving G-action $(\Omega, \mathcal{F}, \mathbb{P}, G)$, where: G is an infinite countable discrete amenable group, $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, and X is a compact metric space associated with Borel σ-algebra $\mathcal{B}_X$.

In our process of building local entropy theory for $\mathbf{F}$, the first and most important step is to prove a local variational principle similar to that given by equations (1.1) and (1.2).

More precisely, let $\mathcal{U}$ be a finite random open cover, f a random continuous function and $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)$, where $\mathcal{P}_\mathbb{P}(\mathcal{E}, G)$ denotes the set of all G-invariant probability measures on $\mathcal{E}$ having the marginal $\mathbb{P}$ over Ω. Denote by $P_\mathcal{E}(f, \mathcal{U}, \mathbf{F})$ and $P_\mathcal{E}(f, \mathbf{F})$ the fiber topological f-pressure of $\mathbf{F}$ with respect to $\mathcal{U}$ and fiber topological f-pressure of $\mathbf{F}$, respectively. Denote by $h_\mu^{(r)}(\mathbf{F}, \mathcal{U})$ and $h_\mu^{(r)}(\mathbf{F})$ the μ-fiber entropy of $\mathbf{F}$ with respect to $\mathcal{U}$ and μ-fiber entropy of $\mathbf{F}$, respectively.

We introduce the property of *factor good* for finite random open covers, and obtain a local variational principle which may be stated as follows:

$$P_\mathcal{E}(f, \mathcal{U}, \mathbf{F}) = \max_{\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E},G)} [h_\mu^{(r)}(\mathbf{F}, \mathcal{U}) + \int_\mathcal{E} f(\omega, x) d\mu(\omega, x)] \tag{1.3}$$

provided that $\mathcal{U}$ is factor good. We show in Theorem 6.10 and Theorem 6.11 that many interesting finite random open covers are factor good.

By taking the supremum over all finite random open covers which are factor good, and using (1.3) one obtains:

$$P_{\mathcal{E}}(f, \mathbf{F}) = \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)], \tag{1.4}$$

which is exactly Kifer's [**46**, Proposition 2.2] in the special case where $G = \mathbb{Z}$. Note that by Remark 7.3, if the underlying G-action $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is trivial, i.e. Ω is a singleton, then the equation (1.3) becomes (1.1) and (1.2), and the equation (1.4) becomes [**59**, Variational Principle 5.2.7], respectively.

In fact, we prove our main result Theorem 7.1 in the more general setting given by a monotone sub-additive invariant family $\mathbf{D}$ of random continuous functions. Denote by $P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F})$ and $P_{\mathcal{E}}(\mathbf{D}, \mathbf{F})$ the fiber topological $\mathbf{D}$-pressure of $\mathbf{F}$ with respect to $\mathcal{U}$ and fiber topological $\mathbf{D}$-pressure of $\mathbf{F}$, respectively. Theorem 7.1 states that: if, in addition, the family $\mathbf{D}$ satisfies the assumption (♠) (cf Chapter 7), then

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})] \tag{1.5}$$

for factor good $\mathcal{U}$, and finally

$$P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}) + \mu(\mathbf{D})]. \tag{1.6}$$

As shown by (7.9) and (7.10), equations (1.5) and (1.6) contain (1.3) and (1.4), respectively. We explore further assumption (♠) in Chapter 9 and Chapter 10. It turns out to be quite natural for countable amenable groups in the following sense: the assumption (♠) always holds if, in addition, either the family $\mathbf{D}$ is strongly sub-additive (cf Proposition 9.1) or the group G is abelian (cf Proposition 10.4).

With the above variational principles, we are able to introduce both topological and measure-theoretic entropy tuples for a continuous bundle random dynamical system, and build a variational relationship between these two kinds of entropy tuples.

It is known (Section 2 of Chapter 13) that the setting of a factor map between topological dynamical systems is in fact equivalent to a special kind of continuous bundle random dynamical systems. Thus, we can apply the above results to study general topological dynamical systems. For example, in Section 3 of Chapter 13 we show that, using (7.5) and (7.6), variants of Theorem 7.1, one can obtain [**51**, Theorem 2.1], the main result of [**51**] by Ledrappier and Walters.

In Section 4 of Chapter 13 , we may apply Theorem 7.1 to generalize the Inner Variational Principle [**23**, Theorem 4] of Downarowicz and Serafin to arbitrary amenable group actions and any finite open cover (cf Theorem 13.2). Theorem 13.2 has also been used to set up symbolic extension theory for amenable group actions by Downarowicz and the second author of the paper [**24**] .

Moreover, our results on entropy tuples of a continuous bundle random dynamical systems, enable us to study entropy tuples for a topological dynamical systems, recovering many recent results in the local entropy theory of $\mathbb{Z}$-actions (cf [**4**, **6**, **29**, **30**, **33**, **35**, **37**]) and of infinite countable discrete amenable group actions (cf [**38**]).

The ideas in the proofs of Propositions 9.1 and 10.4 have been used by Golodets and the authors of the paper to obtain analogues of Kingman's sub-additive ergodic theorem for countable amenable groups ([**19**]).

The paper consists of three parts and is organized as follows.

The first part gives some preliminaries: on infinite countable discrete amenable groups following [**59**, **62**, **69**, **71**], on general measurable dynamical systems of amenable group actions, and on continuous bundle random dynamical systems of an amenable group action extending the case of $\mathbb{Z}$ by [**8**, **46**, **47**, **56**]. In addition to recalling known results, this part contains some new results: firstly, a convergence result (Proposition 2.5) for infinite countable discrete amenable groups extending [**71**, Theorem 5.9] (the difference between Moulin Ollagnier's Proposition 2.3 and our Proposition 2.5 is seen in Example 2.7); secondly, a relative Pinsker formula for a measurable dynamical system with an amenable group action (discussed in [**31**] in the case where the state space is a Lebesgue space), see Theorem 3.4 and Remark 3.5; thirdly, an improved understanding of the local entropy theory of measurable dynamical systems, see Theorem 3.11 and Question 3.12.

In the second part we present and prove our main results. More precisely, in Chapter 5, we take a continuous bundle random dynamical system of an infinite countable discrete amenable group action and a monotone sub-additive invariant family of random continuous functions, and follow the ideas of [**12**, **39**, **64**, **75**] to introduce and discuss the local fiber topological pressure for a finite random open cover. Then in Chapter 6 we introduce and discuss the concept of *factor excellent* and *good* covers, which assumptions are needed for our main result, Theorem 7.1. We show in Theorem 6.10 and Theorem 6.11 that many interesting finite random open covers are factor good. In Chapter 7 we state Theorem 7.1, and give some comments and direct applications. Then, in Chapter 8 we present the proof of Theorem 7.1 following the ideas of [**36**, **38**, **58**, **74**, **75**].

For Theorem 7.1, we need to assume a condition, which we call ($\spadesuit$) on the family of random continuous functions: this is discussed in detail in Chapter 9. In Chapter 10 we discuss the special case of Theorem 7.1 for amenable groups admitting a tiling Følner sequence, and prove that assumption ($\spadesuit$) always holds if the group is abelian. The proof of Theorem 7.1 is for finite random open covers. Inspired by Kifer's work [**46**, §1], in Chapter 11 we generalize Theorem 7.1 to countable random open covers.

The last part of the paper is devoted to applications of the local variational principle established in Part 2. In Chapter 12, following the line of local entropy theory (cf [**30**, Chapter 19] or [**33**]), we introduce and discuss both topological and measure-theoretic entropy tuples for a continuous bundle random dynamical system, and establish a variational relationship between them. Finally, in Chapter 13 we apply the results obtained in the previous chapters to the setting of a general topological dynamical system, incorporating and extending many recent results in the local entropy theory [**4**, **6**, **29**, **30**, **33**, **35**–**38**], as well as establishing (Theorems 13.2 and 13.3) some new variational principles concerning the entropy of a topological dynamical system. We should emphasize that, by the results of [**24**], Theorem 13.2 is important for building the symbolic extension theory of amenable group actions.

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Part 1

Preliminaries

Denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{R}, \mathbb{R}_+$ and $\mathbb{R}_{>0}$ the set of all integers, non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively.

In this part, we give some preliminaries, including: infinite countable discrete amenable groups, measurable dynamical systems, and continuous bundle random dynamical systems.

CHAPTER 2

Infinite countable discrete amenable groups

In this chapter, we recall the principal results from [**59**, **62**, **69**, **71**] and obtain a new convergence result Proposition 2.5 for infinite countable discrete amenable groups. As shown by Remark 2.6, Proposition 2.5 strengthens [**71**, Theorem 5.9] proved by Benjy Weiss. The difference between Moulin Ollagnier's Proposition 2.3 and Proposition 2.5 is demonstrated by Example 2.7; the two results are different even in the setting of an infinite countable discrete amenable group admitting a tiling Følner sequence.

The principal convergence results (Proposition 2.2, Proposition 2.3 and Proposition 2.5) are crucial for the introduction and discussion of local fiber topological pressure of a continuous bundle random dynamical system in Part 2.

Let G be an infinite countable discrete group and denote by e_G the identity of G. Denote by $\mathcal{F}_G$ the set of all non-empty finite subsets of G.

G is called *amenable*, if for each $K \in \mathcal{F}_G$ and any $\delta > 0$ there exists $F \in \mathcal{F}_G$ such that $|F \Delta KF| < \delta |F|$, where $| \bullet |$ is the counting measure of the set $\bullet$, $KF = \{kf : k \in K, f \in F\}$ and $F\Delta KF = (F \setminus KF) \cup (KF \setminus F)$. Let $K \in \mathcal{F}_G$ and $\delta > 0$. Set $K^{-1} = \{k^{-1} : k \in K\}$. $A \in \mathcal{F}_G$ is called (K, δ)*-invariant*, if

$$|K^{-1}A \cap K^{-1}(G \setminus A)| < \delta |A|.$$

A sequence $\{F_n : n \in \mathbb{N}\}$ in $\mathcal{F}_G$ is called a *Følner sequence*, if

$$\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0 \tag{2.1}$$

for each $g \in G$. It is not too hard to obtain the usual asymptotic invariance property from this, viz.: G is amenable if and only if G has a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$. In the class of countable discrete groups, amenable groups include all solvable groups.

In the group $G = \mathbb{Z}$, it is well known that $F_n = \{0, 1, \cdots, n-1\}$ defines a Følner sequence, as, indeed, does $\{a_n, a_n + 1, \cdots, a_n + n - 1\}$ for any sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$.

Standard Assumption 1. *Throughout the current paper, we will assume that G is always an infinite countable discrete amenable group.*

The following terminology and results are due to Ornstein and Weiss [**62**] (see also [**65**, **69**]).

Let $A_1, \cdots, A_k, A \in \mathcal{F}_G$ and $\epsilon \in (0,1)$, $\alpha \in (0,1]$.

(1) Subsets $A_1, \cdots, A_k$ are ϵ*-disjoint* if there are $B_1, \cdots, B_k \in \mathcal{F}_G$ such that

$$B_i \subseteq A_i, \frac{|B_i|}{|A_i|} > 1 - \epsilon \text{ and } B_i \cap B_j = \emptyset \text{ whenever } 1 \le i \ne j \le k.$$

(2) $\{A_1, \cdots, A_k\}$ *α-covers* A if

$$\frac{|A \cap \bigcup_{i=1}^{k} A_i|}{|A|} \geq \alpha.$$

(3) $A_1, \cdots, A_k$ *ε-quasi-tile* A if there exist $C_1, \cdots, C_k \in \mathcal{F}_G$ such that
   (a) for each $i = 1, \cdots, k$, $A_i C_i \subseteq A$ and $\{A_i c : c \in C_i\}$ forms an ϵ-disjoint family,
   (b) $A_i C_i \cap A_j C_j = \emptyset$ if $1 \leq i \neq j \leq k$ and
   (c) $\{A_i C_i : i = 1, \cdots, k\}$ forms a $(1-\epsilon)$-cover of A.

   The subsets $C_1, \cdots, C_k$ are called the *tiling centers*.

We have (see for example [**38**, Proposition 2.3], [**62**] or [**69**, Theorem 2.6]):

PROPOSITION 2.1. *Let $\{F_n : n \in \mathbb{N}\}$ and $\{F'_n : n \in \mathbb{N}\}$ be two Følner sequences of G. Assume that $e_G \in F_1 \subseteq F_2 \subseteq \cdots$. Then for any $\epsilon \in (0, \frac{1}{4})$ and each $N \in \mathbb{N}$, there exist integers $n_1, \cdots, n_k$ with $N \leq n_1 < \cdots < n_k$ such that $F_{n_1}, \cdots, F_{n_k}$ ϵ-quasi-tile F'_m whenever m is sufficiently large.*

It is a well-known fact in analysis that if $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is a sequence satisfying that $a_{n+m} \leq a_n + a_m$ for all $n, m \in \mathbb{N}$, then the sequence $\{\frac{a_n}{n} : n \in \mathbb{N}\}$ converges and

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n} \geq -\infty. \tag{2.2}$$

Similar facts can be proved in the setting of an amenable group as follows.

Let $f : \mathcal{F}_G \to \mathbb{R}$ be a function. Following [**38**], we say that f is:

(1) *monotone*, if $f(E) \leq f(F)$ for any $E, F \in \mathcal{F}_G$ satisfying $E \subseteq F$;
(2) *non-negative*, if $f(F) \geq 0$ for any $F \in \mathcal{F}_G$;
(3) *G-invariant*, if $f(Fg) = f(F)$ for any $F \in \mathcal{F}_G$ and $g \in G$;
(4) *sub-additive*, if $f(E \cup F) \leq f(E) + f(F)$ for any $E, F \in \mathcal{F}_G$.

The following convergence property is well known (see for example [**38**, Lemma 2.4] or [**54**, Theorem 6.1]).

PROPOSITION 2.2. *Let $f : \mathcal{F}_G \to \mathbb{R}$ be a monotone non-negative G-invariant sub-additive function. Then for any Følner sequence $\{F_n : n \in \mathbb{N}\}$ of G, the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and the value of the limit is independent of the choice of the Følner sequence $\{F_n : n \in \mathbb{N}\}$.*

For a function f as in Proposition 2.2, in general we cannot conclude that the limit of the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$ is its infimum. This is shown by Example 2.7 constructed at the end of this chapter (see also Remark 2.8 for more details).

In order to deduce properties analogous to those of (2.2) for the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$, some additional conditions must be added to the assumptions of Proposition 2.2. This can be done in two different ways, both of which will be important for us.

The first extension is:

PROPOSITION 2.3. *Let $f : \mathcal{F}_G \to \mathbb{R}$ be a function. Assume that $f(Eg) = f(E)$ and $f(E \cap F) + f(E \cup F) \leq f(E) + f(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ (we set $f(\emptyset) = 0$ by convention). Then for any Følner sequence $\{F_n : n \in \mathbb{N}\}$ of G, the*