edited by

O. D. ANDERSON

# TIME SERIES ANALYSIS: THEORY AND PRACTICE 2

NORTH-HOLLAND

# TIME SERIES ANALYSIS:

## Theory and Practice 2

Proceedings of the International Conference
held in Dublin, Ireland, March 1982

Edited by

**O.D. ANDERSON**
TSA&F, Nottingham, England

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*To*
*Michael*
*Aine*
*Aoileann*
*Tríona*
*and Colm*
*O hEigeartaigh*

CONTENTS

# WELCOME TO DUBLIN, THE 7TH INTERNATIONAL TIME SERIES MEETING (ITSM) AND THESE PROCEEDINGS

Oliver D. Anderson
*TSA&F, 9 Ingham Grove, Lenton Gardens, Nottingham NG7 2LQ, England*

This Introduction to Time Series Analysis: Theory and Practice 2 provides a brief report on the first Dublin (Ireland) ITSM, together with acknowledgements to all those people who made the volume possible.

## 1. THE EVENT

Fáilte (Welcome)! Details of this Conference (held residentially from 15 to 19 March, 1982, in the hub of Dublin at the Gresham Hotel in O'Connell Street), including the Joining Instructions, Hotel Arrangements, Participants List, Technical Programme and Collection of Abstracts, were given in the February and March 1982 issues of the *TSA&F Flyer*, with some additional (late) abstracts in the April '82 *TSA&F News*.

The Meeting proved to be a most friendly and enjoyable affair, with participants getting inexorably, but pleasantly, pulled into the carnival atmosphere of St Patrick's Week, and being helped to a glimpse of Irish life and culture through the generous and able guidance of Michael and Aine O hEigeartaigh.

Some people managed to squeeze in a visit to the Gate Theatre, which was playing an excellent production of Schaffer's Amadeus; whilst most participants attended the Ceili (Irish Dancing), at the Mansion House, and witnessed other assorted Irish entertainment in traditional Dublin Pubs. A walking tour of central Dublin took in the National Gallery, National Museum and Trinity College (complete with "Book of Kells"); and there was a popular half-day excursion into the nearby Wicklow Mountains.

Winners at the final dinner draw were: Julia Ali (USA) - A free guest place at a 1983 event; Willem Boeschoten (Netherlands) - A free copy of these Conference Proceedings; Jostein Lilleström (Norway) - Time-Series by Sir Maurice Kendall; Jean-Paul Wauters (France) - Payment of his "extras" bill (up to the time of the draw!).

As usual, the Guests (Friends and Family of the Technical Delegates) undoubtedly had a good time.

## 2. THIS BOOK

Due to the fact that rather a lot of invited speakers failed to make it to Dublin, some papers included here were not actually presented in March. Readers should not take this as a precedent, and expect to be able to contribute *in absentia* to future Proceedings volumes.

Papers had to clear a number of hurdles to achieve publication. Some were eliminated at each of the following stages:

(1) Initial Proposal of Tentative Topic;

(2) Provisional Abstract;
(3) Preliminary Screening of Full Paper;
(4) Formal Refereeing of Polished Paper;
(5) Revised Version Assessment;
(6) Final Camera-Ready Preparation;

and, in the end, less than a quarter of the original submissions stayed the course.

Apart from the editing and screening, the reviewing operation involved the consideration of over seventy referee reports; and our warmest thanks are due to all those experts who most kindly helped in the process. These included the following 30 people from 11 countries (which provides some idea of the internationality of our pool of authorities who so generously gave of their time):

Oliver D. Anderson (UK)
Raj J. Bhansali (UK)
Eddie W. Borghers (Belgium)
Christopher Chatfield (UK)
Eivind Damsleth (Norway)
Jan G. de Gooijer (Netherlands)
Frank D.J. Dunstan (UK)
Max Ernoult (France)
Andrew Harvey (UK)
Ruud M.J. Heuts (Netherlands)
Eliot Khabie-Zeitoune (UK)
Tryphon E. Kollintzas (USA)
Johannes Ledolter (USA)
Jostein Lilleströl (Norway)
Helmut Lütkepohl (West Germany)
Guy Mélard (Belgium)
Arnold H.Q.M. Merkies (Netherlands)
Wolfgang Polasek (Austria)
Adrian E. Raftery (Ireland)
George Rawlings (UK)
Brian D. Ripley (UK)
Peter M. Robinson (UK)
Peter Schmidt (USA)
Torsten Söderström (Sweden)
Daniel Sprevak (UK)
Keith D.C. Stoodley (UK)
Stephen J. Taylor (UK)
E.G.F. van Winkel (Netherlands)
A. Morris Walker (UK) &
Walter Wasserfallen (Switzerland).

The preponderance of Western Europeans in the above list is perhaps unfortunate, but reflects very real constraints on whom the ITSM can approach for swift reports, due to the twin problems of less rapid communications and exorbitant air mailing charges, when dealing with referees from farther afield.

No doubt, some critics will find a few papers included which they do not like - and they may be right. But, rather than condemn our very considerable endeavours, might we suggest that they offer to join the pool of referees, and help prevent any similar incidence in the future.

It is also perhaps worth remembering that any markedly adverse criticism of a contribution disagrees with the opinions of at least two (and generally three) other experts (apart from the Editor and Authors), who have refereed the work and recommended publication. One surely risks a label of arrogance in lightly dismissing this general seal of approval. On the whole, we believe that the papers published here will be found as useful, to people concerned with actually analysing time series data, as those appearing in what pass for the better academic journals. Certainly we expect many more practitioners to read them.

Of course, any errors in judgement (as to what should go to press) remain the responsibility solely of the Proceedings Editor - and, for these, we now apologise. Evidently, we have to balance against the evil of mistaken inclusion that of wrongful exclusion. We needs try to be fair to authors as well as readers.

We end the book with biographical sketches for some of the authors - a gesture apparently much appreciated, the last time made (in Forecasting Public Utilities, see the Appendix below), and which we hope to introduce as a regular feature for future Proceedings volumes.

3. OTHER THANKS

The Convenor, Oliver Anderson, is most grateful to all the participants and guests for their attendance and splendid group spirit, and to the speakers for presenting their contributions. As Editor, he is very appreciative of all the hard work put in by both authors and referees, and especially to Dr Bob van Winkel who arranged for the presentation and reviewing of a Dutch session of papers.

And a special word of thanks from everyone to our Irish guides, Michael and Aine O hEigeartaigh.

It remains to thank the reader for his (her) attentions. A good journey to you. Le gach dea-ghui.

APPENDIX

Publications from Earlier Events Organised by Oliver Anderson

The following volumes are published by North-Holland, with O.D. Anderson as Editor:

*Forecasting* (1976 Cambridge Conference) 1979, reprinted 1980. ISBN 0-444-85189-5.
*Time Series* (1979 Nottingham Meeting) 1980. ISBN 0-444-85418-5.
*Analysing Time Series* (1979 Guernsey Meeting) 1980. ISBN 0-444-85464-9.
*Forecasting Public Utilities* (1980 Nottingham Conference) 1980. ISBN 0-444-86046-0.
*Time Series Analysis* (1980 Houston Meeting) 1981. ISBN 0-444-86177-7.
*Time Series Analysis: Theory and Practice 1* (1981 Valencia Meeting) 1982. ISBN 0-444-86337-0.
*Applied Time Series Analysis* (1981 Houston Meeting) 1982. ISBN 0-444-86424-5.

Also, in preparation, we have (with tentative titles):

*Time Series Analysis: Theory and Practice 3* (1982 Valencia Conference)
*Time Series Analysis: Theory and Practice 4* (1982 Cincinnati Meeting).

In all the above, "Meeting" implies an emphasis on Time Series, "Conference" on Forecasting.

REFERENCES

*TSA&F Flyer*. The Time Series Analysis and Forecasting Monthly Information Bulletin, 1980 onwards (ISSN 0260-9053). Edited by O.D. Anderson, 9 Ingham Grove, Lenton Gardens, Nottingham NG7 2LQ, England.

*TSA&F News*. The Time Series Analysis and Forecasting Newsletter (a quarterly publication), 1979 onwards (ISSN 0143-0505). Edited by O.D. Anderson, address as for *Flyer* above.

TIME SERIES ANALYSIS: Theory and Practice 2
O.D. Anderson (editor)



# ESTIMATION FOR MOVING AVERAGE MODELS: WHY DOES IT FAIL ?

Eivind Damsleth

Norwegian Computing Center
P.O.Box 335 Blindern
Oslo 3

Different techniques for estimation of ARIMA-models are investigated, with special emphasis on the moving-average case. It is shown empirically and theoretically that the back-forecasting method suggested by Box & Jenkins (1970) may lead to biased estimates for the MA-parameter(s) as well as for the residual variance. This is particularly so if the series is short and/or the parameters are close to the invertibility region. An explanation for the numerical problems, often encountered in this estimation problem, is also given. The results are of special importance when seasonal models are considered.

## MAIN RESULTS

Three methods for estimation of the MA(1) model are discussed: the conditional sum of squares method, the back-forecasting method (with various numbers of iterations) and the exact maximum likelihood method. In each case the expected value of the optimization criterion is deduced. The Figure on the next page gives one typical example for the behaviour of the function to be minimized by the different methods.

It is seen that the conditional method (0 iterations) tends to give estimates too close to zero. The back-forecasting method, on the other hand, gives estimates which are biased towards the non-invertibility boundary, with the bias increasing with the number of iterations. Finally, the exact maximum likelihood method tends to give unbiased estimates; but, in this case, the function is very flat around the minimum and numerical problems may be encountered.

## SUMMARY

Evidence is provided that the back-forecasting technique may give biased estimates, and also lead to severe numerical problems, if the sample size is small and/or the true parameter value is close to one of the invertibility bounds. Such situations may occur infrequently, but the problem becomes more serious when one is dealing with seasonal models where differencing has been applied to remove a seasonal mean. If the seasonal pattern is stable, the differencing will introduce an MA-parameter close to 1. Further, when seasonal models are considered, it is the number of full periods that enters the formulae for the expected value of the optimization criteria, rather than the number of observations; and this number of full periods will typically be small.

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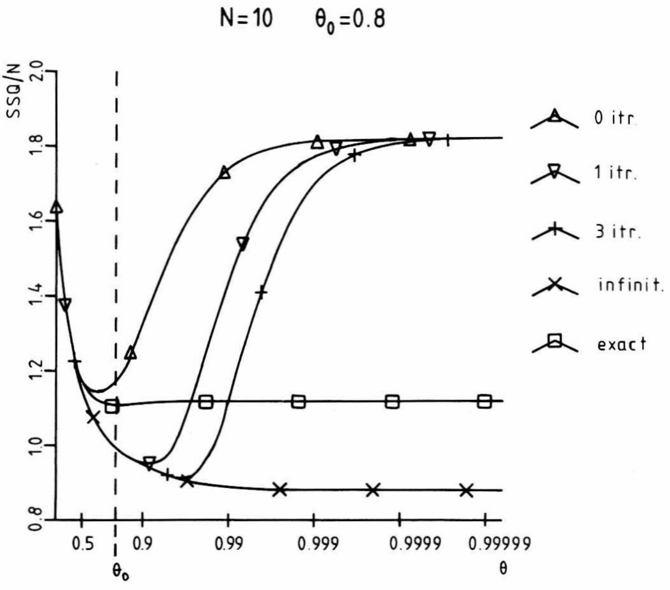
The complete version of this paper is submitted for publication elsewhere. Only the abstract, main results and summary are presented here.




Our evidence adds to the view that exact maximum likelihood methods should be applied when estimating time series models. However, these methods are not as widely used as they should be, since a lot of commercially available program packages offer only the back-forecasting technique. In this case we believe our results show that the back-forecasting should be implemented with only one iteration. The possibility of several iterations may lead to bad estimates as well as severe numerical problems.

REFERENCE

BOX, G.E.P. and JENKINS, G.M. (1970). Time Series Analysis, Forecasting and Control. Holden-Day, San Francisco.

TIME SERIES ANALYSIS: Theory and Practice 2
O.D. Anderson (editor)



# PREDICTION IN CONTINUOUS TIME

E. Khabie-Zeitoune

Mathematics Department
North East London Polytechnic
Romford Rd, London E15

The field of possible applications for a recently introduced modelling methodology is extended. In particular, a method adapted to digital computers, for the prediction of series generated from continuous time processes, is highlighted. As a by-product, special treatment for the two problems of discretisation of continuous time processes and of timewise missing observations becomes unnecessary. Counter examples for the lack of a Wold decomposition, for nonstationary processes in general, are considered and their relation to the specification of a minimum set of unknown parameters stated.

## 1. INTRODUCTION

A general purpose methodology for modelling multiple nonstationary time series has already been introduced in [23] to [27]. The notation and definitions given in these earlier papers will be used here unless otherwise stated.

In addition, T denotes a set, often (though not always) to be interpreted as the time index set. $X_t$, t in T is a stochastic process, real or complex, vector valued, of dimension d. For a discrete (or continuous) vector valued process, T is assumed to be the set of nonnegative integers N (or real numbers $R_+$). When the process is stationary, it is the set of all integers Z ( or all reals R). Other choices of T are possible but will not be considered in this paper. S, a finite subset of T, called the information support, is assumed to be endowed with an order relation, denoted $<$. The elements of S are denoted $t_1, \ldots, t_n$ and correspond to instants when a sample of observations $X_{t_1}, \ldots, X_{t_n}$ are taken. The order relation in S need not necessarily be the one induced by some order relation in T. It may just be any convenient way of enumerating the elements of S.

The methodology of earlier papers is now briefly reviewed. Its mathematical justification arises from the results stated in section 2. Given a model, one defines a method for expressing (in terms of the unknown vector, $\theta$, of model parameters) the process population autocovariances (PACV), $\gamma_{t_i,t_j}$, $t_i$ and $t_j$ in T. The matrix $\gamma_{t_i,t_j}$ is the matrix of variances and covariances of $X_{t_i}$ and $X_{t_j}$. To simplify the notation, it will be written

$\gamma_{i,j}$ unless otherwise indicated. One then restricts attention to the PACV in S, and, for increasing subsets of S, one enters the PACV parameters as block matrix entries in a sequence of matrices $\Gamma_1, \ldots, \Gamma_n$. The matrix $\Gamma_k$ incorporates the PACV parameters for $X_{t_1}, \ldots, X_{t_k}$, k=1,...,n. From these PACV parameters in S, one then computes a triangular system of matrix valued $\phi$-parameters. The last of these parameters, in the discrete time univariate case, reduces to the corresponding classical partial autocorrelation. One then computes another triangular system of matrix valued parameters, and on the basis of these, see (1.1) and (5.1), one introduces a linear transformation from the vectors of the observed sample and defines a new sample $Y_1, \ldots, Y_n$ according to

$$X_i = \sum_{j=1}^{i} \alpha_{i,j} Y_j \quad , \quad i=1,\ldots,n. \tag{1.1}$$

Strictly, one should write $Y_{t_j}$ for $Y_j$, and $\alpha_{t_i,t_j}$ for $\alpha_{i,j}$. The simplified notation of (1.1) will be made use of in this paper although the notation $X_{t_j}$ will be adhered to (rather than $X_j$), as a continual reminder that the sampling time instants may be of irregular frequency.

Under two assumptions, one of which is to be relaxed in this paper, one then proves the fundamental result that $Y_1, \ldots, Y_n$ are uncorrelated. Thus, one can estimate unknown parameters by minimising the sum of square of the Y-vectors with a nonlinear optimisation technique. For Gaussian processes, this minimisation is equivalent to the maximisation of the exact likelihood. This leads to an algorithmic methodology for estimation and prediction, aspects of which were explored in earlier papers. Further aspects will be considered in this paper.

A certain condition made in theorems considered in [26] and [27] is relaxed in section 2, allowing a much wider range of possible applications to be considered.

Classical difficulties in relation to the modelling and prediction of continuous time processes are highlighted in section 3. In contrast, a novel approach is introduced in sections 4, 5 and 6. It is based on the methodology defined earlier and makes a correct use of stochastic differential equations models. A new heuristic for identification is described and is based on blending Monte-Carlo technique with established concepts. A rigorous formulation of the prediction problem in continuous time is highlighted in a manner suitable for digital computing.

As by-products of the above, some further points are discussed. Notwistanding the mathematical results so far obtained in the literature, the two problems of discretisation of continuous time processes and of timewise missing observations, become redundant. This will result from the ability to handle irregular sampling instants. However, the methodology does not purport to provide any answer to the problem of missing values in a multivariate process for which, at a particular instant of time, some (but not all) components are missing. Finally, counter-examples relating to the lack of existence (in general) of a Wold decomposition for nonstationary processes are given. These examples demonstrate the need for considering some autocovariances as unknown parameters, as they may not be computed from structural and white noise parameters. This is in contrast to stationary processes. It also is in contrast with some nonstationary autoregressive processes studied in [14].

Applications to the analysis of physiological data are mentioned in section 7 and some general comments made in section 8.

## 2. SOME GENERAL RESULTS.

The relaxation of a condition for a theorem in [26],[27], namely theorem 2.1 below, as well as the statement of other general results, will be dealt with first in this section. Some counter-examples are then considered in relation to the Wold decomposition and the specification of a minimum set of unknown parameters.

Some notation is first defined. $\gamma_{(k)}$ denotes the block column vector whose block components are $\gamma_{1,k}, \ldots, \gamma_{k,k}$. If $m<k$, $\gamma_{(m),(k)}$ is the column block vector made up of the first m block components of $\gamma_{(k)}$. A similar notational convention is assumed for any other block vector, without further reference. $0_{(k)}$ is a column block vector made up of zeroes. $I_k$ is a matrix made up of k copies of I along the main diagonal, where I is the unit matrix in d dimensions. Matrices considered are partitioned along the last block row and column. $\Psi_k$, $\Lambda_k$, $\Pi_k$ and $\Delta_k$ denote, for each k, the partitioned matrices:

$$\Psi_k = \left[\begin{array}{c|c} I_{k-1} & 0_{(k-1)} \\ \hline 0'_{(k-1)} & \gamma_{k,k}^{-\frac{1}{2}} \end{array}\right], \quad \Lambda_k = \left[\begin{array}{c|c} I_{k-1} & \gamma_{(k-1),(k)}\gamma_{k,k}^{-\frac{1}{2}} \\ \hline 0'_{(k-1)} & I \end{array}\right],$$

$$\Pi_k = \left[\begin{array}{c|c} \Gamma_{k-1} & \gamma_{(k-1),(k)}\gamma_{k,k}^{-\frac{1}{2}} \\ \hline \gamma_{k,k}^{-\frac{1}{2}}\gamma'_{(k-1),(k)} & I \end{array}\right], \quad \Delta_k = \left[\begin{array}{c|c} \delta_{k-1} & 0_{(k-1)} \\ \hline 0'_{(k-1)} & I \end{array}\right],$$

where $\delta_{k-1}$ is defined as

$$\delta_{k-1} = \Gamma_{k-1} - \gamma_{(k-1),(k)}\gamma^{-1}_{k,k}\gamma'_{(k-1),(k)} \,. \tag{2.1}$$

It is then readily checked that

$$\Pi_k = \Lambda_k \Delta_k \Lambda'_k , \quad \Gamma_k = \Psi^{-1}_k \Pi_k \Psi^{-1'}_k = \Psi^{-1}_k \Lambda_k \Delta_k \Lambda'_k \Psi^{-1'}_k \,. \tag{2.2}$$

The Intermediate Variable Autoregression (IVA) parameters are defined as the block entries of $\phi_{(k)}$, where

$$\Gamma_k \phi_{(k)} = \gamma_{(k),(k+1)} , \quad k=1,\ldots,(n-1). \tag{2.3}$$

The last block component $\phi_{k,k}$ of $\phi_{(k)}$ is called the Generalised Partial Autocorrelation (GPA) of order k. As in the case of a univariate weakly stationary process, this parameter reduces to what is classically known as the partial autocorrelation of order k. Again, in the case of a univariate weakly stationary process, the first (k-1) entries of $\gamma(k)$ appear in the classical computation of the partial autocorrelations when using the Levinson-Durbin algorithm, [10], but do not seem to have so far been given any particular name. It is remarked in [15] that these quantities, in the univariate weakly stationary case and up to a multiplicative factor, are playing the role of regression parameters in a series of autoregressive models of increasing order. For this reason, they are called here intermediate variable autoregressions (IVA), reserving more appropriately the label Variable Autoregressions (VA) to related quantities not formally used in this paper. However, a hint is now given as to the definition of the VA quantities. They are the parameters in a linear regression of increasing order of the X-vectors, regressed over previously observed X-vectors, and using the Y-vectors of (1.1) as innovations. In the univariate case, the VA parameters are multiples of the IVA parameters, while in the multivariate case, they are obtained from the IVA parameters by pre-multiplication with specified matrices. It may be noted that in the nonstationary univariate case the GPA coefficients no longer have the classical property (possessed by correlation coefficients) of being between +1 and -1. They however still have the property of being a multiple of the conditional covariance of $X_{t_1}$ and $X_{t_n}$ when intermediate (timewise) variables are maintained fixed. As such, for a not necessarily weakly stationary and not necessarily univariate autoregressive process of order p, the GPA defined through (2.3) are easily checked to be zero from order (p+1) onwards.

The following matrices will also be found useful:

$$
\begin{aligned}
H_k &= \gamma_{k+1,k+1} - \gamma'_{(k),(k+1)}\, \phi_{(k)} \\
&= \gamma_{k+1,k+1} - \phi'_{(k)}\, \gamma_{(k),(k+1)}
\end{aligned}
\qquad (2.4)
$$

$$
L_k = I_k - \phi_{(k)}\, \gamma^{-1}_{k+1,k+1}\, \gamma'_{(k),(k+1)}. \qquad (2.5)
$$

The inversion of $\Gamma_k$ is dealt with next.

<u>Lemma 2.1</u>. If $H^{-1}_{k-1}$ exists, then

(i) $H^{-1}_k = \gamma^{-1}_{k+1,k+1}\,[I + \gamma^{-1}_{k+1,k+1}\, \gamma'_{(k),(k+1)}\, L^{-1}_k\, \phi_{(k)}\,]$

(ii) $L^{-1}_k = I_k + \phi_{(k)}\, H^{-1}_k\, \gamma'_{(k),(k+1)}$

(iii) $\Delta^{-1}_k = \left[\begin{array}{c|c} \delta^{-1}_{k-1} & 0_{(k-1)} \\ \hline 0'_{(k-1)} & I \end{array}\right]$, where

$$
\delta^{-1}_k = L^{-1}_k\, \Gamma^{-1}_k = \Gamma^{-1}_k + \phi_{(k)}\, H^{-1}_k\, \phi'_{(k)} .
$$

Proof. By application of (2.5) and using a formula attributed to Woodbury, see [17].

<u>Lemma 2.2</u>. If $H^{-1}_{k-1}$ and $\Gamma^{-1}_{k-1}$ exist, then for k>1,

$$\Gamma_k^{-1} = \left[\begin{array}{c|c} \Gamma_{k-1}^{-1} & 0\,(k-1) \\ \hline 0'(k-1) & I \end{array}\right] +$$

$$\left[\begin{array}{c|c} \phi(k-1)\, H_{k-1}^{-1}\, \phi'(k-1) & -L_{k-1}^{-1}\, \phi(k-1)\, \gamma_{k,k}^{-1} \\ \hline -\gamma_{k,k}^{-1}\, \phi'(k-1)\, L_{k-1}^{-1} & \gamma_{k,k}^{-1} + \gamma_{k,k}^{-1}\, \gamma'_{(k-1),(k)}\, L_{k-1}^{-1}\, \phi(k-1)\, \gamma_{k,k}^{-1} \end{array}\right]$$

Proof. From lemma 2.1, noticing that

$$\Lambda_k^{-1} = \left[\begin{array}{c|c} I_{k-1} & -\gamma_{(k-1),(k)}\, \gamma_{k,k}^{-\frac{1}{2}} \\ \hline 0'(k-1) & I \end{array}\right]$$

one can then show that $\Gamma^{-1}_{k-1}$ is equal to

$$\left[\begin{array}{c|c} \delta_{k-1}^{-1} & -\delta_{k-1}^{-1}\, \gamma'_{(k-1),(k)}\, \gamma_{k,k}^{-1} \\ \hline -\gamma_{k,k}^{-1}\, \gamma'_{(k-1),(k)}\, \delta_{k-1}^{-1} & \gamma_{k,k}^{-1} + \gamma_{k,k}^{-1}\, \gamma'_{(k-1),(k)}\, \delta_{k-1}^{-1}\, \gamma_{(k-1),(k)}\, \gamma_{k,k}^{-1} \end{array}\right]$$

which leads to the required result.

The statement of the following conditions will be found useful in introducing some generalisation of a theorem in [26],[27].

<u>Condition (A)</u>. The inverse of $H_{k-1}$ and $\Gamma_{k-1}$ exist, for $k=2,\dots,n$.

<u>Condition (B)</u>. $\|\gamma'_{(k-1),(k)}\, \phi(k-1)\| < \|\gamma_{k,k}\|$, the norm of a matrix $A$ being defined as $\|A\| = \sup \|Au\|$, for $\|u\|=1$.

Both conditions were made in [26],[27] for the statement of theorem 2.1 there. Condition (B) is relaxed in theorem 2.1 below.

<u>Theorem 2.1</u>. Assume that only condition (A) is valid. Then, for $k=1,\dots,n$, the inverse of $\Gamma_k$ is given by

$$\left[\begin{array}{c|c} \Gamma_{k-1}^{1} & 0\,(k-1) \\ \hline 0'(k-1) & I \end{array}\right] + \left[\begin{array}{c|c} \phi(k-1)\, H_{k-1}^{-1}\, \phi'(k-1) & -\phi(k-1)\, H_{k-1}^{-1} \\ \hline -H_{k-1}^{-1}\, \phi'(k-1) & H_{k-1}^{-1} \end{array}\right]$$

Proof. First one shows that

$L_{k-1}^{-1}\, \phi(k-1)\, \gamma_{k,k}^{-1} = \phi(k-1)\, H_{k-1}^{-1}$ . Indeed,

$$L_{k-1}^{-1}\, \phi(k-1)\, \gamma_{k,k}^{-1}\, H_{k-1}$$

$$= L_{k-1}^{-1}\, \phi(k-1)\gamma_{k,k}^{-1}\, [\gamma_{k,k} - \gamma'_{(k-1),(k)}\, \phi(k-1)]$$

$$= L_{k-1}^{-1}\, [I_{k-1} - \phi(k-1)\gamma_{k,k}^{-1}\gamma'_{(k-1),(k)}]\, \phi(k-1)$$