Research Problems in Function Theory

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RESEARCH PROBLEMS IN FUNCTION THEORY

by

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编 者 说 明

本书由同一作者的二本小册子和在一本杂志上的一篇文章（作者与其他学者合作）合订成的。

1964 年在伦敦帝国学院召开的经典函数论会议上提出了经典函数论的一些尚未解决的问题。作者按专题归纳，分章顺序排号，编成一个小册子，取名为《函数论中的一些研究问题》，于 1967 年出版。

1973 年在英国的坎特伯雷召开的一次会议上，对 67 年小册子的问题作了部分解答，并提出了若干新问题。作者把从 1964 年以来问题的研究进展情况和提出的新问题按前册形式和顺序，连续编号，以同一书名编成另一小册子出版。

杂志上的一篇文章则和第 2 本小册子类同，收集了 1976 年在英国召开的位势理论和保形映射会议上以及 1975—1976 年度问题讨论会上提出的关于经典函数论的新问题以及前两本小册子的作者的一个报告。

从这三个材料中，我们可以通观 1964—1976 年期间经典函数论的某些问题、进展和概况。全书有以下几个专题：①亚纯函数，②整函数，③调和函数和次调和函数，④多项式，⑤单位园内的函数，⑥单叶函数和多叶函数，⑦其他。各册末均有有关的参考文献。

PREFACE

In this booklet are collected a number of open problems in the classical theory of functions of one complex variable. They have come from a variety of sources. Some are famous problems which have been outstanding for many years, such as Bieberbach's conjecture on the coefficients of schlicht functions (problem 6.1). Others arise out of recent work which has not yet, or only just, been published. Most of them are concerned in some way or other with size, either the order of magnitude of a function or the precise bounds for certain constants. They have been contributed to the author by many friends and in many places, but in particular at the Conference on classical function theory held at Imperial College in September 1964 and so generously supported by N.A.T.O. It is hoped that this collection will serve as a somewhat unorthodox record of that conference and will encourage the researches of the specialist, and inform the nonspecialist of some current trends in an old but lively field of study. I gratefully acknowledge my debt to previous collections of problems and particularly to Erdös [2] and Littlewood [5].

The author of a problem is frequently elusive and I hope I shall be forgiven in the many cases, where no authorship has been attributed. Sometimes references give a good indication and in some cases when I could be sure, I placed the author's name below the problem. May I conclude by thanking all my friends and colleagues who so generously contributed their ideas which have collectively enabled me to write this booklet, the Athlone press, who have been most helpful at every stage, N.A.T.O. for their encouragement and support and Miss Vivien Glover for turning my scribbles into a beautiful typescript.

W. K. H.

CONTENTS

CHAPTER 1

MEROMORPHIC FUNCTIONS AND NEVANLINNA THEORY

We use the usual notation of Nevanlinna theory.[1] If $f(z)$ is meromorphic in $|z| < R$, and $0 < r < R$ we write

$$n(r, a) = n(r, a, f)$$

for the number of roots of the equation $f(z) = a$ in $|z| \leq r$, when multiple roots are counted according to multiplicity and $\bar{n}(r, a)$ when multiple roots are counted only once. We also define

$$N(r, a) = \int_0^r \frac{[n(t, a) - n(0, a)]\, dt}{t} + n(0, a) \log r,$$

$$\bar{N}(r, a) = \int_0^r \frac{[\bar{n}(t, a) - \bar{n}(0, a)]\, dt}{t} + \bar{n}(0, a) \log r,$$

$$m(r, f) = m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta,$$

where $\log^+ x = \max(\log x, 0)$,

$$m(r, a, f) = m\left(r, \infty, \frac{1}{f-a}\right), \quad a \neq \infty.$$

and

$$T(r, f) = m(r, \infty, f) + N(r, \infty, f).$$

Then for every finite a we have by the first fundamental theorem[2]

$$T(r, f) = m(r, a, f) + N(r, a, f) + O(1), \quad \text{as} \quad r \to R. \tag{1.1}$$

We further define the deficiency

$$\delta(a, f) = \varliminf_{r \to R} \frac{m(r, a, f)}{T(r, f)} = 1 - \varlimsup_{r \to R} \frac{N(r, a, f)}{T(r, f)},$$

[1] For an account see e.g. Nevanlinna [1,3] or Hayman [10] which will be denoted by M.F. in the sequel.

[2] M.F., p. 5

the Valiron deficiency

$$\Delta(a,f) = \overline{\lim_{r\to R}} \frac{m(r,a,f)}{T(r,f)},$$

and further

$$\Theta(a,f) = 1 - \overline{\lim_{r\to R}} \frac{\bar{N}(r,a,f)}{T(r,f)}.$$

We then have the second fundamental theorem[1]

$$\sum_a \delta(a,f) < \sum \Theta(a,f) < 2, \tag{1.2}$$

provided that either $R = \infty$ and $f(z)$ is not constant or $R < +\infty$ and

$$\overline{\lim_{r\to R}} \frac{T(r,f)}{\log\{1/(R-r)\}} = +\infty.$$

If $R = +\infty$ we also define the order λ and lower order μ

$$\mu = \underline{\lim_{r\to R}} \frac{\log T(r,f)}{\log r}, \qquad \lambda = \overline{\lim_{r\to R}} \frac{\log T(r,f)}{\log r}.$$

If $\delta(a,f) > 0$ the value a is called deficient. It follows from (1.2) that there are at most countably many deficient values if the conditions for (1.2) are satisfied.

1.1. Is (1.2) all that is true in general? In other words can we construct a meromorphic function $f(z)$ such that $f(z)$ has an arbitrary sequence a_n of deficient values and no others and further that $\delta(a_n, f) = \delta_n$, where δ_n is an arbitrary sequence subject to $\Sigma\delta_n < 2$? (If $f(z)$ is an integral function $\delta(\infty, f) = 1$, so that $\sum_{a\neq\infty} \delta(a,f) < 1$. For a solution of the problem in this case see M.F. p. 80.)

1.2. How big can the set of Valiron deficiencies be for functions in the plane? It is known that

$$N(r,a) = T(r,f) + O\{T(r,f)^{\frac{1}{2}+\epsilon}\} \tag{1.3}$$

as $r \to \infty$, for all a outside a set of inner capacity zero.[2] In case $R < +\infty$ this is more or less best possible but in the plane

[1] M.F., p. 43

[2] Nevanlinna [3], pp. 260–4.

we only know from an example of Valiron [1] that the corresponding set of a can be non-countably infinite. It is also not known whether (1.3) can be sharpened.

1.3. If $f(z)$ is meromorphic of finite order λ and $\sum\delta(a, f) = 2$, it is conjectured that $\lambda = n/2$, where n is an integer and $n \geqslant 2$ and all the deficiences are rational. F. Nevanlinna [1] has proved this result on condition that $f(z)$ has no multiple values so that $n(r, a) = \bar{n}(r, a)$ for every a (see also R. Nevanlinna [2]).

1.4. Let $f(z)$ be an integral function of finite order λ, and let $n_1(r, a)$ denote the number of simple zeros of the equation $f(z) = a$. If

$$n_1(r, a) = O(r^c), \quad n_1(r, b) = O(r^c), \quad \text{as} \quad r \to \infty,$$

where $a \neq b$, $c < \lambda$, is it true that λ is an integral multiple of $\frac{1}{2}$. More strongly is this result true if $\Theta(a) = \frac{1}{2} = \Theta(b)$? (For a somewhat weaker result in this direction see Gol'dberg and Tairova [1].)

1.5. Under what conditions can $\sum\delta(a, f)$ be nearly 2 for an integral function of finite order λ? Pfluger [1] proved that if $\sum\delta(a, f) = 2$, then[1] λ is a positive integer q, $\mu = \lambda$ and all the deficiencies are integral multiples of $1/q$. If further

$$\sum \delta(a, f) > 2-\varepsilon(\mu),$$

where $\varepsilon(\mu)$ depends on μ, then Edrei and Fuchs [1,2] proved that these results remain true 'nearly' in the sense that there exist 'large' deficiencies which are nearly positive integral multiples of $1/q$ and whose sum of deficiencies is 'nearly' 2. Can there be a finite or infinite number of small deficiencies as well in this case?

1.6. N. U. Arakelyan has just proved, (unpublished) that, given $\mu > \frac{1}{2}$ and a countable set E, there exists an integral function $f(z)$ of order μ, for which all the points of E are deficient. Can E be the precise set of deficiencies of f in the sense that f has no other deficient values? It is also conjectured that if a_n are deficient values for an integral function

[1] M.F., p. 115.

of finite order then

$$\sum\{\log[1/\delta(a_n, f)]\}^{-1} < +\infty.$$

(N. U. Arakelyan)

1.7. If $f(z)$ is an integral function of finite order λ which is not an integer it is known that[1]

$$\sum \delta(a, f) \leq 2 - K(\lambda)$$

where $K(\lambda)$ is a positive quantity depending on λ. What is the best possible value for $K(\lambda)$? It is conjectured[2] that if q is the integral part of λ, and if $q \geq 1$, then

$$K(\lambda) = \frac{|\sin(\pi\lambda)|}{q+|\sin(\pi\lambda)|}, \qquad q \leq \lambda < q+\tfrac{1}{2},$$

$$K(\lambda) = \frac{|\sin(\pi\lambda)|}{q+1}, \qquad q+\tfrac{1}{2} \leq \lambda < q+1.$$

This result would be sharp.

If $\lambda \leq \frac{1}{2}$, there are no deficient values, so that $K(\lambda) = 1$. If $\frac{1}{2} < \lambda < 1$ it is known that $K(\lambda) = \sin \pi\lambda$.

1.8. More generally if $f(z)$ is meromorphic in the plane of order λ and $K(\lambda)$ is defined as in 1.7 it is conjectured[1] that for $a \neq b$

$$\overline{\lim_{r \to R}} \frac{N(r, a)+N(r, b)}{T(r, f)} \geq K(\lambda).$$

This is known to be true for $0 < \lambda < 1$. If equality holds in the above inequality it is conjectured that $f(z)$ has regular growth, i.e. $\lambda = \mu$.

1.9. If $f(z)$ is an integral function of finite order λ, which has a finite deficient value find the best possible lower bound for the lower order μ of $f(z)$. (Edrei and Fuchs [1] showed that $\mu > 0$.)

It is also known that for every $\lambda > 1$, $\mu \leq 1$ is possible (A. A. Gol'dberg [1]).

[1] Pfluger [1], M.F., p. 104.
[2] M.F., p. 104, Edrei and Fuchs [1].

1.10. If[1] $f(z)$ is a meromorphic function of finite order with more than two deficient values is it true that if $\sigma > 1$, then

$$\overline{\lim_{r\to\infty}} \frac{T(\sigma r)}{T(r)} < +\infty.$$

1.11. If[1] $f(z)$ is an integral function with at least one finite deficient value does the conclusion of (1.10) hold?

1.12. If $f(z)$ is an integral function of infinite order with real zeros[2] is $\delta(0, f) > 0$? More generally is $\delta(0, f) = 1$?

1.13. If $f(z)$ is an integral function of finite order λ and lower order μ, with real zeros find the best possible bound $B = B(\lambda, \mu)$ such that

$$\delta(0, f) \geqslant B.$$

It is known[2] that $B > 0$ if $1 < \lambda < \infty$, and it is conjectured that $B \to 1$ as $\lambda \to +\infty$.

1.14. If $f(z)$ is meromorphic of finite order then it is known (M.F., pp. 90, 98) that

$$\sum \delta(a, f)^{\alpha}$$

converges if $\alpha > \frac{1}{3}$ but may diverge for every $\alpha < \frac{1}{3}$. What happens when $\alpha = \frac{1}{3}$?

1.15. If $f(z)$ is meromorphic in the plane and of lower order μ and if $\delta = \delta(a, f) > 0$, is it true that, for a sequence

$$r = r_{\nu} \to \infty,$$

$f(z)$ is close to a on a part of the circle $|z| = r_{\nu}$ having angular measure at least

$$\frac{4}{\mu} \sin^{-1} \sqrt{\left(\frac{\delta}{2}\right)} + o(1)?.$$

(For a result in this direction see Edrei [1].)

1.16. For any function $f(z)$ meromorphic in the plane let

$$n(r) = \sup_{a} n(r, a)$$

[1] These problems are suggested by certain conclusions in Edrei and Fuchs [1,2].
[2] Edrei, Fuchs and Hellerstein [1].

be the maximum number of roots of the equation $f(z) = a$ in $|z| < r$, and

$$A(r) = \frac{1}{\pi} \iint_{|z|<r} \frac{|f'(z)|^2}{\{1+|f(z)|^2\}^2} dx\, dy = \frac{1}{\pi} \iint_{|z|<r} \frac{n(r, a)\, |da|^2}{(1+|a|^2)^2}.$$

Then $\pi A(r)$ is the area with due count of multiplicity of the image of the circle $|z| < r$ by $f(z)$ onto the Riemann-sphere and $A(r)$ is the average value of $n(r, a)$ as a moves over the Riemann-sphere. It is known that (M.F., p. 14)

$$1 \leqslant \varlimsup_{r\to\infty} \frac{n(r)}{A(r)} \leqslant e.$$

Can e be replaced by any smaller quantity and in particular by 1?

1.17. For any integral function $f(z)$ of finite order λ in the plane we have

$$1 \leqslant \varliminf_{r\to\infty} \frac{\log M(r, f)}{T(r, f)} \leqslant C(\lambda),$$

where $C(\lambda)$ depends on λ only.[1] It is known[2] that the best possible value of $C(\lambda)$ is $\pi\lambda/\sin(\pi\lambda)$ for $0 < \lambda < \frac{1}{2}$, and it is conjectured that $C(\lambda) = \pi\lambda$ is the corresponding result for $\lambda > \frac{1}{2}$.

1.18. Suppose that $f(z)$ is meromorphic in the plane and that $f(z)$ and $f^{(l)}(z)$ have no zeros for some $l \geqslant 2$. Prove that $f(z) = e^{az+b}$ or $(Az+B)^{-n}$.

[The result is known if $f(z)$ has only a finite number of poles, (Clunie [3], M.F., p. 67) or if $f(z)$ has finite order and $f \neq 0$, $f' \neq 0, f'' \neq 0$, and

$$\varliminf_{r\to\infty} \frac{\log n(r, f)}{\log r} < +\infty.$$

(Hayman [8]) or if none of the derivatives of $f(z)$ have any zeros and $f(z)$ has unrestricted growth. (Pólya [1], M.F., p. 63.)]

[1] This follows very simply from M.F., Theorem 1.6, p. 18.
[2] Wahlund [1].

1.19. Suppose that $f(z)$ is meromorphic in the plane and

$$f'(z)f(z)^n \neq 1, \qquad \text{where} \quad n \geqslant 1.$$

Prove that $f(z)$ is constant. This is known to be true for $n \geqslant 3$. (Hayman [8].)

1.20. If $f(z)$ is non-constant and meromorphic in the plane and $n = 3$ or 4 prove that $\phi(z) = f'(z) - f(z)^n$ assumes all finite complex values. This is known to be true if $f(z)$ is an integral function or if $n \geqslant 5$ if $f(z)$ is meromorphic. (ibid.)

In this connection it would be most interesting to have general conditions under which a polynomial in $f(z)$ and its derivatives can fail to take some complex value. Especially when $f(z)$ is meromorphic rather than an integral function rather little is known (see however Clunie [3,4] and M.F., ch. 3).

1.21. If $F(z)$ is non-constant in the plane it is known (M.F., pp. 55–6) that

$$\alpha_f = \overline{\lim_{r\to\infty}} \frac{T(r,f)}{T(r,f')} \geqslant \begin{cases} \frac{1}{2} \text{ if } f(z) \text{ is meromorphic;} \\ 1, \text{ if } f(z) \text{ is an integral function.} \end{cases}$$

These inequalities are sharp. It is not known whether

$$\beta_f = \underline{\lim_{r\to\infty}} \frac{T(r,f)}{T(r,f')}$$

can be greater than one or even infinite. It is known that β_f is finite if $f(z)$ has finite order. Examples show that α_f may be infinite for integral functions of any order μ, i.e. $0 \leqslant \mu \leqslant \infty$, and that given any positive constants K, μ there exists an integral function of order at most μ such that

$$\frac{T(r,f)}{T(r,f')} > K$$

on a set of r having positive lower logarithmic density. (For this and related results see Hayman [11].)

1.22. The second fundamental Theorem is a consequence of the inequality (M.F., formula (2.9), p. 43)

$$\sum_{\nu=1}^{k} \bar{N}(r, a_\nu, f) \geqslant [q-2+o(1)]T(r,f) \tag{1.4}$$

which holds for any $q \geqslant 3$ distinct numbers a_ν, as $r \to \infty$ outside a set E of finite measure, if $f(z)$ is meromorphic in the plane. The exceptional set E is known to be unnecessary if $f(z)$ has finite order. Does (1.4) also hold as $r \to \infty$ without restriction if $f(z)$ has infinite order?

1.23. Under what circumstances does $f(z_0+z)$ have the same deficiencies as $f(z)$? It was shown by Dugué [1] that this need not be the case for meromorphic functions and by Hayman [4] that it is not necessarily true for integral functions of infinite order. The case of functions of finite order remains open. Valiron [2] notes that a sufficient condition is that

$$\frac{T(r+1,f)}{T(r,f)} \to 1, \qquad \text{as} \quad r \to \infty,$$

and this is the case in particular if $\lambda - \mu < 1$, where λ is the order and μ the lower order. Since for integral functions of order $\mu \leqslant \frac{1}{2}$ there are no deficiencies anyway it follows that the result is true at any rate for integral functions of order $\lambda < \frac{3}{2}$ and, since $\mu \geqslant 0$ always, for meromorphic functions of order less than one.

CHAPTER 2

INTEGRAL FUNCTIONS

Asymptotic values and paths

Let $f(z)$ be an integral function. We say that a is an asymptotic value of $f(z)$, if

$$f(z) \to a,$$

as $z \to \infty$ along a path Γ, called a corresponding asymptotic path. Some of the most interesting open problems concerning integral functions centre on these asymptotic values and paths. It follows from a famous result of Ahlfors [1] that an integral function of finite order k can have at most $2k$ distinct finite asymptotic values. On the other hand, by a theorem of Iversen [1] ∞ is an asymptotic value of every integral function. Some of the following problems are concerned with generalizations arising out of the above two theorems. Throughout this section

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|$$

denotes the maximum modulus of $f(z)$.

2.1. Suppose that $f(z)$ is an integral function of finite order. What can we say about the set E of values w such that

$$\text{(i)} \quad \mu(r, f-w) = \min_{|z|=r} |f(z)-w| \to 0, \qquad \text{as} \quad r \to \infty;$$

or

$$\text{(ii)} \quad m\left(r, \frac{1}{f-w}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta})-w}\right| d\theta \to \infty?$$

Clearly (ii) implies (i). By the result of Arakelyan quoted for problem 1.6 the set of deficient values, which is clearly contained in E, can include any countable set. Can E be non-countably infinite in case (ii) or contain interior points in case (i)?

2.2. Produce a general method for constructing an integral function of finite order and in fact minimal growth which tends to different asymptotic values $w_1, w_2 \dots w_k$ as $z \to \infty$ along preassigned asymptotic paths $C_1, C_2 \dots C_k$. (Known methods Kennedy [1], Al-Katifi [1], only seem to work if the w_ν are all equal, unless the C_ν are straight lines.)

2.3. If $\phi(z)$ is an integral function growing slowly compared with the function $f(z)$, we can consider $\phi(z)$ to be an asymptotic function of $f(z)$ if $f(z)-\phi(z) \to 0$ as $z \to \infty$ along a path Γ. Is it true that an integral function of order k can have almost $2k$ distinct asymptotic functions of order less than $\frac{1}{2}$? (If

$$f-\phi_1(z) \to 0, \qquad f-\phi_2(z) \to 0$$

along the same path Γ and $\phi_1(z)$, $\phi_2(z)$ have order less than $\frac{1}{2}$ then by Wiman's theorem $\phi_1(z) \equiv \phi_2(z)$.) A positive result in this direction is due to Denjoy [1], but only when the paths are straight lines. The result when the $\phi_\nu(z)$ are polynomials is true (and a trivial consequence of Ahlfors' theorem for asymptotic values).

2.4. Suppose that $f(z)$ is a meromorphic function in the plane, and that for some number θ, $0 \leqslant \theta < 2\pi$, the function $f(z)$ assumes infinitely often every value with at most two exceptions in every angle $\theta-\varepsilon < \arg z < \theta+\varepsilon$, when $\varepsilon > 0$. Then the ray $\arg z = \theta$ is called a 'Julia line'. It is known[1] that if $f(z)$ is an integral function or if $f(z)$ is meromorphic and

$$\overline{\lim_{r\to\infty}} \frac{T(r,f)}{(\log r)^2} = +\infty,$$

(but not necessarily otherwise,) at least one direction of Julia exists. What can we say about the exceptional values at different Julia lines? In particular can an integral function $f(z)$ have one exceptional (finite) value a at one Julia line Γ_a and a different exceptional value b at a different Julia line Γ_b?

(C. Renyi)

2.5. What can we say about the set E of values a which an integral function $f(z)$ assumes infinitely often in every angle?

[1] See Lehto [1].