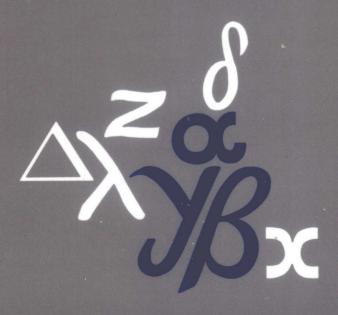
ALGEBRAIC STRUCTURE AND MATRICES

VOLUME 2 OF ADVANCED ALGEBRA

E. A. Maxwell



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ALGEBRA



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AND MATRICES

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BEING PART II OF ADVANCED ALGEBRA



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PREFACE

Besides the general acknowledgement mentioned in the Introduction, I would also express my more specific thanks to Mr A. P. Rollett and Dr G. Matthews who gave me very valuable help and criticism at the manuscript stage. Mr W. G. Kellaway, of the Department of Education in the University of Cambridge, read the proofs and made a number of important suggestions.

This manuscript set the printers a difficult task, and I am most grateful to them for the skill and cheerfulness with which they have overcome it.

E. A. M.

March, 1964

It is probably true that this would have been a different book if written in 1969—I have learned much as the years pass, even though they have been comparatively few. But I doubt whether the basic plan would have changed greatly. This still seems to me to be a reasonable way to present the subject to the still immature, but developing, young mathematician.

I am grateful for many comments and much advice. To adopt all would have gone beyond second printing to second edition, but I have incorporated improvements where readily possible. There were also more mis-prints (or, to give them their proper names, mistakes) than I like in the first printing and I acknowledge help from a number of friends, especially the late Mr. C. V. Durell and, in kindly but persistent detail, Mr. F. Gerrish.

E. A. M.

February, 1969

INTRODUCTION

Since the peaceful days of 1958-9 when I wrote the first volume of this work, an explosion of considerable violence has struck the world of teachers of mathematics. 'In 1959, the Organisation for European Economic Cooperation convoked at Royaumont, near Paris, France, a two-week seminar on "New Thinking in School Mathematics".' This quotation, from the later O.E.E.C. book, Synopses for Modern Secondary School Mathematics, describes the force that gathered in preparation for the explosion. Much preliminary work had been done before 1959, but it is fair to ascribe to that gathering and to its chairman, Professor Marshall Stone of the University of Chicago, high credit for the more directed activity that has since taken place.

For my own part, I had the privilege of being present for some of the time, and of speaking and discussing the problems informally. It became clear to me that something vital was involved, though I did not feel able to go as far as many of the protagonists would wish. Whether or not I have judged correctly here, the reader must decide.

It is not possible, I think, to write at this stage a fully satisfactory book on 'modern algebra' for class-room use; there just is not sufficient teaching experience of the problems involved. I have tried to look with the eye of imagination at a beginner approaching the subject, and to judge how he can best be served as he tries, on the one hand, to see what it is all about and, on the other (should he be proceeding to further study) to prepare himself for a firm grasp of the details that will come later.

I have had in mind a pupil in the top forms at school or in the first year of a university course. In writing, I have done all I can to make myself clear, but I have not attempted to pretend that the subject is easy. It is exciting and illuminating, and very rewarding, but it is not easy, and the pupil who wishes to master it will have to make the effort.

The choice of subject-matter has been extremely difficult; many alternatives are possible both in content and in presentation. Every reader familiar with the subject as a whole will regret some omission or other. A conspicuous omission, for example, is any reference to mathematical logic. Notation, too, has proved troublesome, for many alternatives are in existence. I can only hope that the particular choices made will not prove unduly troublesome in later reading; my consolation is that other variants would probably have proved no less unpopular.

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It would be hard to express my indebtedness for the material used. I have given a fairly detailed bibliography and take this opportunity of recording how much I owe to all these authors. The list is far from exhaustive, but it does indicate possible next steps for reading in a subject which I hope this book may do something to encourage.

BIBLIOGRAPHY

The following list of books for further, or alternative, reading includes many which I have consulted and to which I would pay tribute. It also contains references to exploratory work both before and after the conference at Royaumont.

1. The introduction of 'modern mathematics' into the curriculum

Beberman, M., An emerging programme of secondary school mathematics. Oxford, 1958.

An account of the Illinois experiment dating from 1952, one of the first serious attempts to tackle the problem.

Program for College preparatory mathematics (with appendices). 1958.

Report of a Commission on Mathematics made to the College Entrance Examination Board of America. Set up in 1955, it gives a complete 'modernization' programme for all school levels, with class-room details.

New thinking in school mathematics. 1961.

The report of the O.E.E.C. Conference at Royaumont.

School mathematics in O.E.E.C. countries.

A summary of the position in schools and training colleges for all O.E.E.C. countries.

Mathematics for physicists and engineers.

Lectures and discussions at an O.E.E.C. seminar.

Synopses for modern secondary school mathematics.

A very detailed series of suggestions for a radical re-thinking of secondary school mathematics, with considerable elaboration of syllabuses.

Report of the Gulbenkian Foundation on mathematics at A-level. 1960.

The first detailed study of the problem in the United Kingdom to suggest practical steps for class-room and examination.

On teaching mathematics. Pergamon, 1961.

A symposium held at Southampton under Professor Thwaites, with, again, detailed suggestions for syllabuses.

2. Textbooks at a fairly elementary level

Adler, I., The new mathematics. 1958. A 'Mentor' book; New American Library of World Literature, Inc.

Covers a wide range vividly and simply. The guiding theme is the gradual development and abstraction of the number system.

Alexandroff, P. S., *Introduction to the theory of Groups*. Blackie, 1959. Translation of a fascinating account by a Russian master of the subject.

Coxeter, H. S. M., *Introduction to geometry*. Wiley, 1961. Not strictly relevant, but an original and thrilling account of similar possibilities in geometry. BIBLIOGRAPHY XV

Davenport, H., *The higher arithmetic*. Hutchinson, 1952. An admirable survey, indispensable for the mathematician.

Goodstein, R. L., Fundamental concepts of mathematics. Pergamon, 1962. A simple but scholarly review for 'the cultivated amateur'. Excellent chapters on 'Classes and truth functions' and 'Networks and maps'.

Ledermann, W., Introduction to the theory of finite groups. Oliver and Boyd, 1949.

A very thorough elementary account.

Mansfield, D. E. and Thompson, D., Mathematics, a new approach. Chatto and Windus, 1962.

An important, and highly successful, attempt to produce a textbook at a really elementary level. One of the most original and lucid books yet produced to deal with the problem. It is based on actual class-room experience dating from the 'pre-Royaumont' era, and deals with the 'O' level of the General Certificate of Education.

Sawyer, W. W., A concrete approach to abstract algebra. Freeman (San Francisco), 1959.

Covers a very wide range in a thoroughly readable manner.

3. More advanced texts

Aitken, A. C., Determinants and matrices. Oliver and Boyd, 1939.

An extraordinarily able piece of compression, and an excellent account of

the essentials. The present author is indebted to his treatment of linear systems of equations.

Albert, A. A., Fundamental concepts of higher algebra. Chicago, 1956.

A very thorough review, especially aimed towards the theory of finite fields.

Birkhoff, G. and MacLane, S., A survey of modern algebra. Macmillan, 1941. A very widely ranging survey of the subject as a whole.

Ferrar, W. L., Algebra. Oxford, 1941.

A well-established standard textbook which covers the basic groundwork very readably.

Hadley, G., Linear algebra. Addison-Wesley, 1961.

A thorough introduction at a comparatively elementary level. It 'leads logically to its author's *Linear Programming* text'.

Halmos, P. R., Finite-dimensional vector spaces. Princeton, 1948. An abstract, but clear and lucid, account.

Hohn, F. E., Applied Boolean algebra, an elementary introduction. Macmillan, New York, 1960.

Deals with Boolean algebra, relay circuits, and propositional logic.

Hodge, W. V. D. and Pedoe, D., Methods of algebraic geometry. Book I. Cambridg 1947.

Although concerned with geometry, Book I contains a very detailed introduction to the algebraic essentials.

Mirsky, L., An introduction to linear algebra. Oxford, 1955.

Covers those parts of the subject which must form part of any standard course.

Shilov, G. E., Theory of linear spaces. Prentice-Hall, 1961.

An excellent translation of a first-rate account of the subject which will be enjoyed by all mathematicians.

Thrall, R. M. and Tornheim, L., Vector spaces and matrices. Chapman and Hall, 1957.

Most of the standard theory is covered, with a wealth of detail. An excellent text for the advanced student.

Turnbull, H. W. and Aitken, A. C., Theory of Cannonical matrices. Blackie, 1932.

A very elegant account of the basic principles.

Van der Waerden, B. L., Modern algebra. Ungar. 1949.

A two-volume work which has already established itself as a classic. Very readable, but, of necessity, not easy.

4. Foreign Language texts

Structures algébriques et structures topologiques. By various authors, being No. 7 of Monographies de l'Enseignement Mathématique. Paris, 1958.

A brilliant symposium, with all the lucidity that would be expected.

Elements de Mathématique. Bourbaki, 1954 etc. I. Théorie des ensembles, II Algèbre.

Perhaps the most famous work of present-day mathematics by a group of authors believed to be commutative.

Mathématiques du 20e siècle. Brussels, 1960.

Various authors, following a seminar. Gives a good insight into some modern trends.

Godement, R., Cours d'algèbre. Hermann, 1963.

A masterly exposition, essential for the advanced student. At 664 pages for 60 francs it is expensive but extremely good value.

5. Supplementary list

(Texts issued while this book was in press.)

G. Matthews, Matrices I; C. A. R. Bailey, Sets and logic, I; F. B. Lovis, Computers I; J. A. C. Reynolds, Shape, size and place. These are St. Dunston's College Booklets, published by Edward Arnold.

A. J. Moakes, Numerical Mathematics and The core of Mathematics; W. Chellingsworth, Mathematics for circuits. These are in the series, Introductory Monographs in Mathematics, published by Macmillan.

Georges Papy, Groups. Macmillan, 1964.

T. J. Fletcher, Editor, Some lessons in mathematics. Cambridge University Press. 1964.

The School Mathematics Project, 'Book T'. Cambridge University Press, 1964.

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SECTION I

ALGEBRAIC STRUCTURE