LECTURE NOTES IN PHYSICS

58

Computing Methods in Applied Sciences

Edited by

R. Glowinski and J. L. Lions

Lecture Notes in Physics

Edited by J. Ehlers, München, K. Hepp, Zürich,
H. A. Weidenmüller, Heidelberg, and J. Zittartz, Köln
Managing Editor: W. Beiglböck, Heidelberg

58

Computing Methods in Applied Sciences

Second International Symposium
December 15–19, 1975

IRIA LABORIA
Institut de Recherche d'Informatique
et d'Automatique

Edited by R. Glowinski and J. L. Lions

Springer-Verlag
Berlin · Heidelberg · New York 1976

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Domaine de Voluceau BP 105
Rocquencourt
78150 Le Chesnay/France

ISBN 3-540-08003-1 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-08003-1 Springer-Verlag New York · Heidelberg · Berlin

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.

INTRODUCTION

This book contains part of the lectures which were presented during the Second International Symposium on Computing Methods in Applied Sciences and Engineering, December 15 to December 19, 1975, organised by IRIA-LABORIA under the sponsorship of IFIP, AFCET and GAMNI.

More than 400 scientists and engineers from many countries attended this meeting.

The interest evidenced within the scientific community prompted IRIA to organise such a meeting every two years, evolving topics to fit the developments of science and techniques. With this goal in mind the next event in this series will take place from December 5 to December 9, 1977.

The organizers wish to express their gratitude to Mr. A. DANZIN, Director of IRIA and address their thanks to each session chairman who directed very interesting discussions and also to all the speakers.

Sincere gratitude is also expressed to the IRIA Public Relations Office whose help contributed greatly to the success of this Symposium.

The remainder of these proceedings are published as Lecture Notes in Economics and Mathematical Systems, Volume 134.

R. GLOWINSKI J.L. LIONS

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INTRODUCTION

Le présent volume rassemble une partie des travaux présentés au Colloque International sur les "Méthodes de Calcul Scientifique et Technique" organisé par l'IRIA-LABORIA du 15 au 19 Décembre 1975, sous le patronage de l'I.F.I.P., de l'A.F.C.E.T. et du G.A.M.N.I.

Ce Colloque a réuni à Versailles près de 400 chercheurs et ingénieurs de toutes nationalités.

Devant l'intérêt suscité dans des milieux scientifique variés, l'IRIA a décidé d'organiser tous les deux ans, à une époque semblable de l'année, une réunion de type analogue - naturellement avec une évolution des sujets adaptée au développement de la Science et des techniques. La prochaine manifestation de cette série aura donc lieu du 5 au 9 Décembre 1977.

Les organisateurs remercient Monsieur A. DANZIN, Directeur de l'IRIA et les divers Présidents de séance qui ont animé d'intéressantes discussions ainsi que tous les conférenciers qui ont pris part à ce Colloque.

Nos remerciements vont également au Service des Relations Extérieurs de l'IRIA dont l'aide a joué un rôle essentiel dans l'organisation de cette rencontre.

L'autre partie de ce Colloque est publiée sous Lecture Notes in Economics and Mathematical Systems, Volume 134.

R. GLOWINSKI J.L. LIONS

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GÉNÉRALITÉS
GENERALITIES

A SIMPLE THEORY OF GEOMETRICAL STIFFNESS WITH APPLICATIONS TO BEAM AND SHELL PROBLEMS

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SUMMARY

Geometrical stiffness is the basis for any attempt to study the behaviour of slender beams and thin shells under conditions in which large deflections may occur with small strains. Not all problems require high accuracy in the representation of the geometrical stiffness. These are generally certain self-equilibrating stress systems (natural modes) which are the principal contributors to the geometrical stiffness. In particular, stress systems which produce rigid body moments due to rigid body rotations of the element are generally most important. Also, very great differences in bending stiffness about different axes may make it necessary to consider otherwise unimportant natural forces.

Although beams are considered among the simplest of structural elements their analysis when bent and twisted in three dimensions is by no means simple and the same is true of the consideration of their geometrical stiffness in space. Thus the beam in space may be considered as a test case for the general methods developed here.

Large deflection theory of plate and shells is generally concerned with deflections of the order of the thickness which are sufficient to induce considerable membrane stresses. Thus the non-linear effect arises from the induced membrane stresses rather than from gross changes in geometry. The problem of snap through and the perhaps rather academic problem of the three dimensional elastica pose some very difficult finite element applications in which the geometry changes are of the order of the structural dimensions. To tackle such problems using a highly sophisticated shell element such as SHEBA is not an easy undertaking. For other more immediately practical reasons it has been necessary to develop a simple flat facet shell element with transverse shear deformation also. This element, which is a displacement but not a Rayleigh-Ritz element, has only 18 nodal freedoms and is adaptable to thin, thick and sandwich type applications, is especially suitable for large deflection problems.

The paper presents some large deflection examples for beams and it is hoped also to have ready some non-trivial applications to shells.

1. Introduction

A paper presented at the first IRIA conference [1] developed a relatively simple theory of large strain in membranes and solid bodies. In compact solid bodies large displacements are necessarily associated with large strains so that a separation of the non-linear effects of large material strains and large geometry changes is not possible. Thin rods and shells are special cases of solid bodies in which large displacements may take place even when the strains remain small. It is clear that a comprehensive theory of large strains with large displacements embraces that of large displacements with small strains as a special case. Such a theory could indeed be used to provide a check on any large displacement small strain theory. For engineering applications it would seem that a large displacement theory restricted ab initio to small strains must be simpler and therefore more economical than the more general theory.

The basic assumption of the theory to be given here is that within each finite element the small displacement stress strain relations are valid. For this reason the natural mode method is especially applicable. Stress and strain are here supposed to apply in their generalised sense and may, for example, include bending moments and curvatures. Some previous work [2,3] partially developed the theory and more recently [4] an attempt was made to simplify the treatment of geometrical stiffness and to elucidate some difficulties arising because of asymmetry due to the use of rotational degrees of freedom. An alternative treatment, in which apparently no asymmetries are observed, is due to Besseling [5] .

It is found that the treatment of geometrical stiffness is easier for natural modes depending only on nodal displacements (translations). The stiffness matrix is always symmetrical in this case. When nodal rotations or higher order nodal parameters are used the geometrical stiffness matrices are asymmetrical. In spite of this the assembled geometrical stiffness may be symmetrical and in many cases the same result may be obtained by mathematically symmetrizing the element geometrical stiffnesses before assembly. This will be so in elements with translational and rotational nodal freedoms if the nodal moments are always applied as semi-tangential torques - that is a torque represented by two equal forces pairs acting at the ends of a cross rigidly connected to the node. If the nodal moments are not applied in this way they will give rise to displacement dependent moments which will form an additional geometrical stiffness. This latter geometrical stiffness will be symmetrical when the applied loads are conservative but otherwise asymmetrical.

Although the theory is applicable to any finite element with nodal forces and moments its application in practice is mainly to slender beams and thin shells. The slender beam with much greater stiffness about one principal axis bent and twisted in space is one of the most difficult non-linear problems in structural mechanics. It provides a test case of more than academic interest for the present theory and also poses some very sensitive numerical problems.

On the other hand the problem of large deflections in shells is not so difficult from the geometrical stiffness point of view. This is because the only geometrical stiffness of importance is that arising from the membrane stresses. The high precision shell elements such as SHEBA are in principle applicable to large deflection problems in shells. So far as is known to the authors the considerable soft-ware investment, required to calculate a three-dimensional elastica and problems of similar difficulty, has not yet been made. For other reasons [4] it was necessary at ISD to develop a simple engineering accuracy plate and shell element with transverse shear deformation and since this element involves only translation and rotation freedoms it seemed a natural candidate to attempt large displacement shell problems. The element is a plane facet type triangle constructed by a combination of Rayleigh-Ritz and physical lumping methods. Some small displacement examples of the application of this element were given in [4] . In this paper some large displacement examples are given.

2. Elements with only translational nodal freedoms.

This case is simpler than the general one and includes most practical membrane and solid elements such as the TRIM and TET class. Such elements are characterised by the fact that the nodal freedoms $\boldsymbol{\rho}$ are vectors which are increments of global position vectors $\boldsymbol{r}$. Thus,

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{\rho} \tag{1}$$

$$\boldsymbol{r}_\Delta = \boldsymbol{\rho}_\Delta \tag{2}$$

Note that the deformation of the element may be written in equivalent ways as

$$\boldsymbol{\rho}_N = \boldsymbol{\rho}_N(\boldsymbol{r}) = \boldsymbol{\rho}_N(\boldsymbol{\rho}) \tag{3}$$

where $\boldsymbol{\rho}_N$ is the vector of natural modes. For those not familiar with the idea of natural modes we may give the example in which they are the extensions of the sides in the TRIM 3 or TET 4 elements [2 , 3] . Thus the natural modes are not unique or complete but are always equal in number to the difference of the element nodal freedoms and the rigid body freedoms. Also in the present context of small strain large displacement theory the natural modes are small even when the global displacements are large. For a finite element analysis, including geometrical stiffness, we require the increment of $\boldsymbol{\rho}_N$ up to the second order in the increment of $\boldsymbol{\rho}$. For a particular natural mode ρ_{Np}, the increment,

$$\rho_{Np\Delta} = \frac{\partial \rho_{Np}}{\partial \boldsymbol{\rho}} \boldsymbol{\rho}_\Delta + \frac{1}{2} \boldsymbol{\rho}_\Delta^t \frac{\partial^2 \rho_{Np}}{\partial \boldsymbol{\rho} \partial \boldsymbol{\rho}^t} \boldsymbol{\rho}_\Delta \tag{4}$$

From the first expression on the right of (4) we have, to first order accuracy,

$$\boldsymbol{\rho}_{N\Delta} = \boldsymbol{a}_N \, \boldsymbol{\rho}_{\Delta} \tag{5}$$

where,

$$\boldsymbol{a}_N = \left\{ \boldsymbol{a}_{N1} \; \cdots \; \boldsymbol{a}_{Np} \; \cdots \; \boldsymbol{a}_{N\nu} \right\} = \frac{\partial \boldsymbol{\rho}_N}{\partial \boldsymbol{\rho}} \tag{6}$$

and

$$\boldsymbol{a}_{Np} = \frac{\partial \boldsymbol{\rho}_{Np}}{\partial \boldsymbol{\rho}} \tag{7}$$

The matrix $\boldsymbol{a}_N$ depends on $\boldsymbol{\rho}$ and for an increment $\boldsymbol{\rho}_{\Delta}$, $\boldsymbol{a}_N$ becomes

$$\boldsymbol{a}_N + \boldsymbol{a}_{N\Delta} \tag{8}$$

where from (4),

$$\boldsymbol{a}_{Np\Delta} = \boldsymbol{\rho}_{\Delta}^{t} \, \frac{\partial^2 \boldsymbol{\rho}_{Np}}{\partial \boldsymbol{\rho} \, \partial \boldsymbol{\rho}^{t}} \tag{9}$$

Note that equation (4) may be written as

$$\boldsymbol{\rho}_{N\Delta} = \left[\boldsymbol{a}_N + \frac{1}{2} \boldsymbol{a}_{N\Delta} \right] \boldsymbol{\rho}_{\Delta} \tag{10}$$

This equation is useful when $\boldsymbol{\rho}_N$ is accumulated from the increments instead of using the more accurate up-dating from equation (3). The matrices $\boldsymbol{a}_N$ and $\boldsymbol{a}_{N\Delta}$ are the only properties of an element required, in addition to the natural stiffness, to carry out a complete large displacement small strain calculation.

However, in elements with a large number of nodal freedoms it may be sufficiently accurate to include only some of the natural modes in $\mathbf{a}_{N\Delta}$ and even to consider only the rigid body components of $\boldsymbol{\rho}$ as contributing to $\mathbf{a}_{N\Delta}$. For the latter purpose we have to define also the rigid body modes $\boldsymbol{\rho}$ which together with $\boldsymbol{\rho}_N$ form the vector

$$\boldsymbol{\rho}' = \{ \boldsymbol{\rho}_0 \ \boldsymbol{\rho}_N \} = \boldsymbol{\rho}'(\boldsymbol{\rho}) \tag{11}$$

In the three dimensional case this is valid only for small values of the rigid body rotations in $\boldsymbol{\rho}_0$

As an example of the formation of the rigid body movements from the global displacements we take the TRIM 3 element. It will be assumed that we require a relation of the form

$$\boldsymbol{\rho}'_\Delta = \mathbf{a} \ \boldsymbol{\rho}_\Delta = \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_N \end{bmatrix} \boldsymbol{\rho}_\Delta \tag{12}$$

To have a consistent relation between the cartesian expression for small rotations and the rigid body rotation ρ_{03} one must have

$$\rho_{03} = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \tag{13}$$

where u, v are the cartesian displacements with respect to orthogonal axes X, Y . The linear displacement field in the TRIM 3 is

$$\left. \begin{aligned} u &= p_0 + p_1 x + p_2 y \\ v &= q_0 + q_1 x + q_2 y \end{aligned} \right\} \tag{14}$$

Then if the origin of X, Y is at the centroid and u_1, v_1 etc. are the nodal displacements,

$$\left.\begin{array}{l} u_1 + u_2 + u_3 = 3p_o \\ v_1 + v_2 + v_3 = 3q_o \end{array}\right\} \qquad (15)$$

and hence

$$\rho_{01} = p_o = \frac{1}{3}\left[u_1 + u_2 + u_3 \right]$$

$$\rho_{02} = q_o = \frac{1}{3}\left[v_1 + v_2 + v_3 \right] \qquad (16)$$

From (13)

$$\rho_{\cdot 3} = \frac{1}{2}\left(q_1 - p_2 \right) \qquad (17)$$

q_1, p_2 may be found from equations (14) written for each corner. The result is,

$$\rho_{03} = \frac{1}{4\Omega}\left(x_{23}u_1 + y_{23}v_1 + x_{31}u_2 + y_{31}v_2 + x_{12}u_3 + y_{12}v_3 \right) \qquad (18)$$

where

Ω = area of triangle.

Thus the matrix $\boldsymbol{a}_0$ associated with

$$\boldsymbol{\rho} = \left\{ u_1 \; v_1 \; u_2 \; v_2 \; u_3 \; v_3 \right\} \qquad (19)$$

is

$$\boldsymbol{a}_0 = \begin{bmatrix} 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ -\frac{x_{23}}{4\Omega} & -\frac{y_{23}}{4\Omega} & -\frac{x_{31}}{4\Omega} & -\frac{y_{31}}{4\Omega} & -\frac{x_{12}}{4\Omega} & -\frac{y_{12}}{4\Omega} \end{bmatrix} \tag{20}$$

Remembering that the natural modes are the elongations of the sides the matrix $\boldsymbol{a}_N$ is,

$$\boldsymbol{a}_N = \boldsymbol{l}^{-1} \begin{bmatrix} 0 & 0 & -x_{23} & -y_{23} & x_{23} & y_{23} \\ x_{31} & y_{31} & 0 & 0 & -x_{31} & y_{31} \\ x_{12} & y_{12} & -x_{12} & y_{12} & 0 & 0 \end{bmatrix} \tag{21}$$

where

$$\boldsymbol{l} = \lceil l_{23} \quad l_{31} \quad l_{12} \rfloor \tag{22}$$

Inversion of $\boldsymbol{a}$ now gives

$$\boldsymbol{A}_e = [\boldsymbol{A}_0 \quad \boldsymbol{A}_N] \tag{23}$$

in which

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \\ 1 & 0 & -y_3 \\ 0 & 1 & x_3 \end{bmatrix} \tag{24}$$