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Preface

The concept of fractional derivative appeared for the first time in a famous correspondence between G.A. de L'Hospital and G.W. Leibniz, in 1695. Many mathematicians have further developed this area and we can mention the studies of L. Euler (1730), J.L. Lagrange (1772), P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823), J. Liouville (1832), B. Riemann (1847), H.L. Greer (1859), H. Holmgren (1865), A.K. Grünwald (1867), A.V. Letnikov (1868), N.Ya. Sonin (1869), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917), H. Weyl (1919), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924), A. Zygmund (1935), E.R. Love (1938), A. Erdélyi (1939), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949) and W. Feller (1952). In the past sixty years, fractional calculus had played a very important role in various fields such as physics, chemistry, mechanics, electricity, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc.

In the last decade, fractional calculus has been recognized as one of the best tools to describe long-memory processes. Such models are interesting for engineers and physicists but also for pure mathematicians. The most important among such models are those described by differential equations containing fractional-order derivatives. Their evolutions behave in a much more complex way than in the classical integer-order case and the study of the corresponding theory is a hugely demanding task. Although some results of qualitative analysis for fractional differential equations can be similarly obtained, many classical methods are hardly applicable directly to fractional differential equations. New theories and methods are thus required to be specifically developed, whose investigation becomes more challenging. Comparing with classical theory of differential equations, the researches on the theory of fractional differential equations are only on their initial stage of development.

This monograph is devoted to a rapidly developing area of the research for the qualitative theory of fractional differential equations. In particular, we are interested in the basic theory of fractional differential equations. Such basic theory should be the starting point for further research concerning the dynamics, control, numerical analysis and applications of fractional differential equations. The book is divided into six chapters. Chapter 1 introduces preliminary facts from fractional calculus, nonlinear analysis and semigroup theory. In Chapter 2, we present a unified framework to investigate the basic existence theory for discontinuous fractional functional differential equations with bounded delay, unbounded delay and infinite delay. Chapter 3 is devoted to the study of fractional differential equations in Banach spaces via measure of noncompactness method, topological degree method and Picard operator technique. In Chapter 4, we first present some techniques for the investigation of fractional evolution equations governed by C_0-semigroup, then we discuss fractional evolution equations with almost sectorial operators. In Chapter 5, by using critical point theory, we give a new approach to study boundary value problems of fractional differential equations. And in the last chapter, we present recent advances on theory for fractional partial differential equations including fractional Euler-Lagrange equations, time-fractional diffusion equations, fractional Hamiltonian systems and fractional Schrödinger equations.

The material in this monograph are based on the research work carried out by the author and other experts during the past four years. The book is self-contained and unified in presentation, and it provides the necessary background material required to go further into the subject and explore the rich research literature. Each chapter concludes with a section devoted to notes and bibliographical remarks and all abstract results are illustrated by examples. The tools used include many classical and modern nonlinear analysis methods. This book is useful for researchers and graduate students for research, seminars, and advanced graduate courses, in pure and applied mathematics, physics, mechanics, engineering, biology, and related disciplines.

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Yong Zhou
October 2013, Xiangtan, China

Contents

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we introduce some notations and basic facts on fractional calculus, nonlinear analysis and semigroup which are needed throughout this book.

1.2 Some Notations, Concepts and Lemmas

As usual $\mathbb{N}$ denotes the set of positive integer numbers and $\mathbb{N}_0$ the set of nonnegative integer numbers. $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+$ denotes the set of nonnegative reals and $\mathbb{R}^+$ the set of positive reals. Let $\mathbb{C}$ be the set of complex numbers.

We recall that a vector space X equipped with a norm $|\cdot|$ is called a normed vector space. A subset E of a normed vector space X is said to be bounded if there exists a number K such that $|x| \leq K$ for all $x \in E$. A subset E of a normed vector space X is called convex if for any $x, y \in E$, $ax + (1-a)y \in E$ for all $a \in [0, 1]$.

A sequence $\{x_n\}$ in a normed vector space X is said to converge to the vector x in X if and only if the sequence $\{|x_n - x|\}$ converges to zero as $n \to \infty$. A sequence $\{x_n\}$ in a normed vector space X is called a Cauchy sequence if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for all $n, m \geq N(\varepsilon)$, $|x_n - x_m| < \varepsilon$. Clearly a convergent sequence is also a Cauchy sequence, but the converse may not be true. A space X where every Cauchy sequence of elements of X converges to an element of X is called a complete space. A complete normed vector space is said to be a Banach space.

Let E be a subset of a Banach space X. A point $x \in X$ is said to be a limit point of E if there exists a sequence of vectors in E which converges to x. We say a subset E is closed if E contains all of its limit points. The union of E and its limit points is called the closure of E and will be denoted by $\bar{E}$. Let X, F be normed vector spaces, and E be a subset of X. An operator $\mathscr{T} : E \to F$ is continuous at a point $x \in E$ if and only if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\mathscr{T}x - \mathscr{T}y| < \varepsilon$ for all $y \in E$ with $|x - y| < \delta$. Further, $\mathscr{T}$ is continuous on E, or simply continuous, if it is continuous at all points of E.

We say that a subset E of a Banach space X is compact if every sequence of vectors in E contains a subsequence which converges to a vector in E. We say that E is relatively compact in X if every sequence of vectors in E contains a subsequence which converges to a vector in X, i.e., E is relatively compact in X if $\bar{E}$ is compact.

Let $J = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval of $\mathbb{R}$. We assume that X is a Banach space with the norm $|\cdot|$. Denote $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm

$$\|x\| = \sup_{t \in J} |x(t)|,$$

where $x \in C(J, X)$. $C^n(J, X)$ $(n \in \mathbb{N}_0)$ denotes the set of mappings having n times continuously differentiable on J, $AC(J, X)$ is the space of functions which are absolutely continuous on J and $AC^n(J, X)$ $(n \in \mathbb{N}_0)$ is the space of functions f such that $f \in C^{n-1}(J, X)$ and $f^{(n-1)} \in AC(J, X)$. In particular, $AC^1(J, X) = AC(J, X)$.

Let $1 \leq p \leq \infty$. $L^p(J, X)$ denotes the Banach space of all measurable functions $f : J \to X$. $L^p(J, X)$ is normed by

$$\|f\|_{L^p J} = \begin{cases} \left(\int_J |f(t)|^p dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \{ \sup\limits_{t \in J \setminus \bar{J}} |f(t)| \}, & p = \infty. \end{cases}$$

In particular, $L^1(J, X)$ is the Banach space of measurable functions $f : J \to X$ with the norm

$$\|f\|_{LJ} = \int_J |f(t)| dt,$$

and $L^\infty(J, X)$ is the Banach space of measurable functions $f : J \to X$ which are bounded, equipped with the norm

$$\|f\|_{L^\infty J} = \inf\{c > 0 : |f(t)| \leq c, \text{ a.e. } t \in J\}.$$

Lemma 1.1 (Hölder inequality). Assume that $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(J, X), g \in L^q(J, X)$, then for $1 \leq p \leq \infty$, $fg \in L^1(J, X)$ and

$$\|fg\|_{LJ} \leq \|f\|_{L^p J} \|g\|_{L^q J}.$$

A family F in $C(J, X)$ is called uniformly bounded if there exists a positive constant K such that $|f(t)| \leq K$ for all $t \in J$ and all $f \in F$. Further, F is called equicontinuous, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ for all $t_1, t_2 \in J$ with $|t_1 - t_2| < \delta$ and all $f \in F$.

Lemma 1.2 (Arzela-Ascoli's theorem). If a family $F = \{f(t)\}$ in $C(J, \mathbb{R})$ is uniformly bounded and equicontinuous on J, then F has a uniformly convergent subsequence $\{f_n(t)\}_{n=1}^{\infty}$. If a family $F = \{f(t)\}$ in $C(J, X)$ is uniformly bounded and equicontinuous on J, and for any $t^* \in J$, $\{f(t^*)\}$ is relatively compact, then F has a uniformly convergent subsequence $\{f_n(t)\}_{n=1}^{\infty}$.

The Arzela-Ascoli's Theorem is the key to the following result: A subset F in $C(J,\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on J.

Lemma 1.3 (Lebesgue's dominated convergence theorem). Let E be a measurable set and let $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in E, and for every $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ a.e. in E, where g is integrable on E. Then

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Finally, we state the Bochner's theorem.

Lemma 1.4 (Bochner's theorem). A measurable function $f : (a,b) \to X$ is Bochner integrable if $|f|$ is Lebesgue integrable.

1.3 Fractional Calculus

The gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (Re(z) > 0),$$

where $t^{z-1} = e^{(z-1)\log(t)}$. This integral is convergent for all complex $z \in \mathbb{C}$ $(Re(z) > 0)$.

For this function the reduction formula

$$\Gamma(z+1) = z\Gamma(z) \qquad (Re(z) > 0)$$

holds. In particular, if $z = n \in \mathbb{N}_0$, then

$$\Gamma(n+1) = n! \qquad (n \in \mathbb{N}_0)$$

with (as usual) $0! = 1$.

Let us consider some of the starting points for a discussion of fractional calculus. One development begins with a generalization of repeated integration. Thus if f is locally integrable on (c,∞), then the n-fold iterated integral is given by

$$\begin{aligned} {}_cD_t^{-n} f(t) &= \int_c^t ds_1 \int_c^{s_1} ds_2 \cdots \int_c^{s_{n-1}} f(s_n)ds_n \\ &= \frac{1}{(n-1)!}\int_c^t (t-s)^{n-1} f(s)ds \end{aligned}$$

for almost all t with $-\infty \leq c < t < \infty$ and $n \in \mathbb{N}$. Writing $(n-1)! = \Gamma(n)$, an immediate generalization is the integral of f of fractional order $\alpha > 0$,

$${}_cD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1} f(s)ds \quad \text{(left hand)}$$

and similarly for $-\infty < t < d \leq \infty$

$${}_tD_d^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^d (s-t)^{\alpha-1} f(s)ds \quad \text{(right hand)}$$

both being defined for suitable f.

A number of definitions for the fractional derivative has emerged over the years, we refer the reader to Diethelm, 2010; Hilfer, 2006; Kilbas, Srivastava and Trujillo, 2006; Miller and Ross, 1993; Podlubny, 1999. In this book, we restrict our attention to the use of the Riemann-Liouville and Caputo fractional derivatives. In this section, we introduce some basic definitions and properties of the fractional integrals and fractional derivatives which are used further in this book. The materials in this section are taken from Kilbas, Srivastava and Trujillo, 2006.

1.3.1 *Definitions*

Definition 1.5 (Left and right Riemann-Liouville fractional integrals). Let $J = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval of $\mathbb{R}$. The left and right Riemann-Liouville fractional integrals ${}_aD_t^{-\alpha} f(t)$ and ${}_tD_b^{-\alpha} f(t)$ of order $\alpha \in \mathbb{R}^+$, are defined by

$$
{}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a, \quad \alpha > 0 \tag{1.1}
$$

and

$$
{}_tD_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds, \quad t < b, \quad \alpha > 0, \tag{1.2}
$$

respectively, provided the right-hand sides are pointwise defined on $[a, b]$. When $\alpha = n \in \mathbb{N}$, the definitions (1.1) and (1.2) coincide with the n-th integrals of the form

$$
{}_aD_t^{-n} f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)ds
$$

and

$$
{}_tD_b^{-n} f(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} f(s)ds.
$$

Definition 1.6 (Left and right Riemann-Liouville fractional derivatives). The left and right Riemann-Liouville fractional derivatives ${}_aD_t^{\alpha} f(t)$ and ${}_tD_b^{\alpha} f(t)$ of order $\alpha \in \mathbb{R}_+$, are defined by

$$
\begin{aligned}
{}_aD_t^{\alpha} f(t) &= \frac{d^n}{dt^n} {}_aD_t^{-(n-\alpha)} f(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left(\int_a^t (t-s)^{n-\alpha-1} f(s)ds \right), \quad t > a
\end{aligned} \tag{1.3}
$$

and

$$
\begin{aligned}
{}_tD_b^{\alpha} f(t) &= (-1)^n \frac{d^n}{dt^n} {}_tD_b^{-(n-\alpha)} f(t) \\
&= \frac{1}{\Gamma(n-\alpha)} (-1)^n \frac{d^n}{dt^n} \left(\int_t^b (s-t)^{n-\alpha-1} f(s)ds \right), \quad t < b,
\end{aligned} \tag{1.4}
$$

respectively, where $n = [\alpha] + 1$, $[\alpha]$ means the integer part of α. In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$_aD_t^0 f(t) = {}_tD_b^0 f(t) = f(t),$$

$$_aD_t^n f(t) = f^{(n)}(t) \quad \text{and} \quad {}_tD_b^n f(t) = (-1)^n f^{(n)}(t),$$

where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n. If $0 < \alpha < 1$, then

$$_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_a^t (t-s)^{-\alpha} f(s) ds \right), \quad t > a$$

and

$$_tD_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_t^b (s-t)^{-\alpha} f(s) ds \right), \quad t < b.$$

Remark 1.7. If $f \in C([a,b], \mathbb{R}^N)$, it is obvious that Riemann-Liouville fractional integral of order $\alpha > 0$ exists on $[a,b]$. On the other hand, following Lemma 2.2 in Kilbas, Srivastava and Trujillo, 2006, we know that the Riemann-Liouville fractional derivative of order $\alpha \in [n-1, n)$ exists almost everywhere on $[a,b]$ if $f \in AC^n([a,b], \mathbb{R}^N)$.

The left and right Caputo fractional derivatives are defined via above Riemann-Liouville fractional derivatives.

Definition 1.8 (Left and right Caputo fractional derivatives). The left and right Caputo fractional derivatives ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ of order $\alpha \in \mathbb{R}_+$ are defined by

$${}_a^C D_t^\alpha f(t) = {}_aD_t^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] \tag{1.5}$$

and

$${}_t^C D_b^\alpha f(t) = {}_tD_b^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (b-t)^k \right], \tag{1.6}$$

respectively, where

$$n = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}_0; \; n = \alpha \text{ for } \alpha \in \mathbb{N}_0. \tag{1.7}$$

In particular, when $0 < \alpha < 1$, then

$${}_a^C D_t^\alpha f(t) = {}_aD_t^\alpha (f(t) - f(a))$$

and

$${}_t^C D_b^\alpha f(t) = {}_tD_b^\alpha (f(t) - f(b)).$$

The Riemann-Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations.

Property 1.9.

(i) If $\alpha \notin \mathbb{N}_0$ and $f(t)$ is a function for which the Caputo fractional derivatives ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ of order $\alpha \in \mathbb{R}^+$ exist together with the Riemann-Liouville fractional derivatives ${}_a D_t^\alpha f(t)$ and ${}_t D_b^\alpha f(t)$, then

$$ {}_a^C D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha} $$

and

$$ {}_t^C D_b^\alpha f(t) = {}_t D_b^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}, $$

where $n = [\alpha] + 1$. In particular, when $0 < \alpha < 1$, we have

$$ {}_a^C D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} $$

and

$$ {}_t^C D_b^\alpha f(t) = {}_t D_b^\alpha f(t) - \frac{f(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}. $$

(ii) If $\alpha = n \in \mathbb{N}_0$ and the usual derivative $f^{(n)}(t)$ of order n exists, then ${}_a^C D_t^n f(t)$ and ${}_t^C D_b^n f(t)$ are represented by

$$ {}_a^C D_t^n f(t) = f^{(n)}(t) \quad \text{and} \quad {}_t^C D_b^n f(t) = (-1)^n f^{(n)}(t). \tag{1.8} $$

Property 1.10. Let $\alpha \in \mathbb{R}_+$ and let n be given by (1.7). If $f \in AC^n([a,b], \mathbb{R}^N)$, then the Caputo fractional derivatives ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ exist almost everywhere on $[a,b]$.

(i) If $\alpha \notin \mathbb{N}_0$, ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ are represented by

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds\right) \tag{1.9} $$

and

$$ {}_t^C D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)ds\right) \tag{1.10} $$

respectively, where $n = [\alpha] + 1$. In particular, when $0 < \alpha < 1$ and $f \in AC([a,b], \mathbb{R}^N)$,

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha} f'(s)ds\right) \tag{1.11} $$

and

$$ {}_t^C D_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)}\left(\int_t^b (s-t)^{-\alpha} f'(s)ds\right). \tag{1.12} $$

(ii) If $\alpha = n \in \mathbb{N}_0$ then ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ are represented by (1.8). In particular,

$$ {}_a^C D_t^0 f(t) = {}_t^C D_b^0 f(t) = f(t). $$