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$$\delta(u) := \int_0^T u(t)\delta W(t) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \qquad t \in [0, T]$$

$$E\Big[F \int_0^T u(t)\delta W(t)\Big] = E\Big[\int_0^T u(t) D_t F dt\Big]$$

$$= \int_0^T D_t u(s)\delta W(s) - u(t)$$

Giulia Di Nunno
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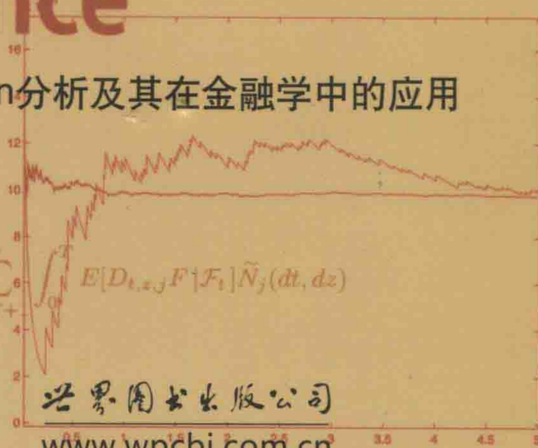
Frank Proske

Malliavin Calculus for Lévy Processes with Applications to Finance

Lévy过程的Malliavin分析及其在金融学中的应用



$$F = E[F] + \sum_{j=1}^{N} \int_0^T E[D_{t,j}F \mid \mathcal{F}_t] dW_j(t) + \sum_{j=N+1}^{L} \int_0^T E[D_{t,z,j}F \mid \mathcal{F}_t] \tilde{N}_j(dt, dz)$$

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To Christian. To my parents.
G.D.N.

To Eva.
B.Ø.

To Simone, Paulina and Siegfried.
F.P.

Preface

There are already several excellent books on Malliavin calculus. However, most of them deal only with the theory of Malliavin calculus for Brownian motion, with [35] as an honorable exception. Moreover, most of them discuss only the application to regularity results for solutions of SDEs, as this was the original motivation when Paul Malliavin introduced the infinite-dimensional calculus in 1978 in [158]. In the recent years, Malliavin calculus has found many applications in stochastic control and within finance. At the same time, Lévy processes have become important in financial modeling. In view of this, we have seen the need for a book that deals with Malliavin calculus for Lévy processes in general, not just Brownian motion, and that presents some of the most important and recent applications to finance.

It is the purpose of this book to try to fill this need. In this monograph we present a general Malliavin calculus for Lévy processes, covering both the Brownian motion case and the pure jump martingale case via Poisson random measures, and also some combination of the two. We also present many of the recent applications to finance, including the following:

- The Clark–Ocone theorem and hedging formulae
- Minimal variance hedging in incomplete markets
- Sensitivity analysis results and efficient computation of the "greeks"
- Optimal portfolio with partial information
- Optimal portfolio in an anticipating environment
- Optimal consumption in a general information setting
- Insider trading

To be able to handle these applications, we develop a general theory of anticipative stochastic calculus for Lévy processes involving the Malliavin derivative, the Skorohod integral, the forward integral, which were originally introduced for the Brownian setting only. We dedicate some chapters to the generalization of our results to the white noise framework, which often turns out to be a suitable setting for the theory. Moreover, this enables us to prove results that are general enough for the financial applications, for example, the generalized Clark–Ocone theorem.

This book is based on a series of courses that we have given in different years and to different audiences. The first one was given at the Norwegian School of Economics and Business Administration (NHH) in Bergen in 1996, at that time about Brownian motion only. Other courses were held later, every time including more updated material. In particular, we mention the courses given at the Department of Mathematics and at the Center of Mathematics for Applications (CMA) at the University of Oslo and also the intensive or compact courses presented at the University of Ulm in July 2006, at the University of Cape Town in December 2006, at the Indian Institute of Science (IIS) in Bangalore in January 2007, and at the Nanyang Technological University in Singapore in January 2008.

At all these occasions we met engaged students and attentive readers. We thank all of them for their active participation to the classes and their feedback. Our work has benefitted from the collaboration and useful comments from many people, including Fred Espen Benth, Maximilian Josef Butz, Delphine David, Inga Baadshaug Eide, Xavier Gabaix, Martin Groth, Yaozhong Hu, Asma Khedher, Paul Kettler, An Ta Thi Kieu, Jørgen Sjaastad, Thilo Meyer-Brandis, Farai Julius Mhlanga, Yeliz Yolcu Okur, Olivier Menoukeu Pamen, Ulrich Rieder, Goncalo Reiss, Steffen Sjursen, Alexander Sokol, Agnès Sulem, Olli Wallin, Diane Wilcox, Frank Wittemann, Mihail Zervos, Tusheng Zhang, and Xunyu Zhou. We thank them all for their help. Our special thanks go to Paul Malliavin for the inspiration and continuous encouragement he has given us throughout the time we have worked on this book. We also acknowledge with gratitude the technical support with computers of the Drift-IT at the Department of Mathematics at the University of Oslo.

Oslo,
June 2008.

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Bernt Øksendal
Frank Proske

Contents

Introduction

The mathematical theory now known as *Malliavin calculus* was first introduced by Paul Malliavin in [158] as an infinite-dimensional integration by parts technique. The purpose of this calculus was to prove the results about the smoothness of densities of solutions of stochastic differential equations driven by Brownian motion. For several years this was the only known application. Therefore, since this theory was considered quite complicated by many, Malliavin calculus remained a relatively unknown theory also among mathematicians for some time. Many mathematicians simply considered the theory as too difficult when compared with the results it produced. Moreover, to a large extent, these results could also be obtained by using Hörmander's earlier theory on hypoelliptic operators. See also, for example, [20, 114, 225, 230].

This was the situation until 1984, when Ocone in [173] obtained an explicit interpretation of the Clark representation formula [46, 47] in terms of the Malliavin derivative. This remarkable result later became known as the *Clark–Ocone formula*. Sometimes also called *Clark–Haussmann–Ocone formula* in view of the contribution of Haussmann in 1979, see [98]. In 1991, Ocone and Karatzas [174] applied this result to finance. They proved that the Clark–Ocone formula can be used to obtain explicit formulae for replicating portfolios of contingent claims in complete markets.

Since then, new literature helped to distribute these results to a wider audience, both among mathematicians and researchers in finance. See, for example, the monographs [53, 161, 169, 212, 216] and the introductory lecture notes [178]; see also [206].

The next breakthrough came in 1999, when Fournié et al. [81] obtained numerically tractable formulae for the computation of the so-called *greeks* in finance, also known as *parameters of sensitivity*. In the recent years, many new applications of the Malliavin calculus have been found, including partial information optimal control, insider trading and, more generally, anticipative stochastic calculus.

At the same time Malliavin calculus was extended from the original setting of Brownian motion to more general Lévy processes. This extensions were at first motivated by and taylored to the original application within the study of smoothness of densities (see e.g. [12, 35, 37, 38, 44, 141, 142, 143, 163, 189, 190, 218, 219]) and then developed largely targeting the applications to finance, where Lévy processes based models are now widely used (see, e.g., [25, 29, 64, 69, 148, 171, 181]). Within this last direction, some extension to random fields of Lévy type has also been developed, see, for example, [61, 62]. Other extension of Malliavin calculus within quantum probability have also appeared, see, for example, [84, 85].

One way of interpreting the Malliavin derivative of a given random variable $F = F(\omega)$, $\omega \in \Omega$, on the given probability space $(\Omega, \mathcal{F}, P)$ is to regard it as a derivative with respect to the random parameter ω. For this to make sense, one needs some mathematical structure on the space Ω. In the original approach used by Malliavin, for the Brownian motion case, Ω is represented as the *Wiener space* $C_0([0,T])$ of continuous functions $\omega : [0,T] \longrightarrow \mathbb{R}$ with $\omega(0) = 0$, equipped with the uniform topology. In this book we prefer to use the representation of Hida [99], namely to represent Ω as the *space $\mathcal{S}'$ of tempered distributions* $\omega : \mathcal{S} \longrightarrow \mathbb{R}$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ (see Chap. 5). The corresponding probability measure P is constructed by means of the Bochner–Minlos theorem. This is a classical setting of white noise theory. This approach has the advantage that the Malliavin derivative D_tF of a random variable $F : \mathcal{S}' \longrightarrow \mathbb{R}$ can simply be regarded as a *stochastic gradient.*

In fact, if γ is deterministic and in $L^2(\mathbb{R})$ (note that $L^2(\mathbb{R}) \subset \mathcal{S}'$), we define the *directional derivative of F in the direction* γ, $D_\gamma F$, as follows:

$$D_\gamma F(\omega) = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon}, \qquad \omega \in \mathcal{S}',$$

if the limit exists in $L^2(P)$. If there exists a process $\Psi(\omega, t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$D_\gamma F(\omega) = \int_{\mathbb{R}} \Psi(\omega, t)\gamma(t)dt, \quad \omega \in \mathcal{S}'$$

for all $\gamma \in L^2(\mathbb{R})$, then we say that F is *Malliavin–Hida differentiable* and we define

$$D_tF(\omega) := \Psi(\omega, t), \qquad \omega \in \mathcal{S}'$$

as the *Malliavin–(Hida) derivative* (or *stochastic gradient*) of F at t.

This gives a simple and intuitive interpretation of the Malliavin derivative in the Brownian motion case. Moreover, some of the basic properties of calculus such as *chain rule* follow easily from this definition. See Chap. 6.

Alternatively, the Malliavin derivative can also be introduced by means of the *Wiener–Itô chaos expansion* [120]:

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$