Calculus
A Short Course with Applications

Gerald Freilich
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CALCULUS

A Short Course with Applications

SECOND EDITION

PREFACE

This textbook offers an intuitive approach to calculus with a strong and creative emphasis on applications. It is written for the student whose mathematical background may include only a working knowledge of high-school algebra. Our goal is to give students a working knowledge of calculus as well as an awareness of its important applications in today's world.

The Second Edition of *Calculus: A Short Course with Applications* incorporates improvements based on five years of classroom experience at New York University and Queens College (CUNY) as well as improvements suggested by reviewers, colleagues, and readers. Although the NYU course evolved to meet the needs of students in the management sciences, we are mindful of the needs of students in the biological and social sciences and have included a wide range of applications in these areas. The revisions for this edition have been extensive, and we are hopeful they will prove beneficial to students and instructors alike. In this Second Edition, we have

- Doubled the number of exercises
- Created many new figures and examples
- Highlighted major concepts with vivid color type
- Emphasized with color the graphic statements of more than 260 figures
- Discussed concepts in an applied context
- Compiled a checklist of key topics at the end of each chapter
- Assembled a full set of review exercises to complete each chapter

Answers to the even-numbered exercises appear at the end of the text, with complete solutions to those exercises available in a Solutions Manual.

The basic philosophy of the book remains the same: well-motivated discussion presented in a clear, lively way. Here are some of the features we emphasize as we present our treatment of calculus—in a compact 390 pages.

•**Applications** *Theoretical concepts make sense when developed in an understandable applied context.* Realistic applications are woven into the development of all major topics, and are illustrated by ample sets of examples and exercises. The stage is set early (in Section 1.3) with a self-contained account of terminology from the management sciences that enhances the review material in Chapter 1. Many important topics are introduced with simplified case studies that lead into routine worked examples and exercises. These case studies are labeled "Illustration" in the text.

•**Examples and exercises** *Examples should be structured to develop mathematical skills and an intuitive feel for theoretical concepts.* Every new topic begins with routine problems, worked out in complete detail. These are followed by applied problems with a more verbal orientation. We take pains to give applied examples a realistic feeling, so students become accustomed to seeing the mathematical problem emerge from a description in words. Common sense dictates that this be done gradually, paving the way for full-scale word problems in optimization. In every chapter, exercises are arranged in the same order as the topics are discussed, to make assignments easier.

•**Relevance** *The course should prepare students to deal with calculus in their chosen disciplines.* Several features are designed to meet the real needs of the management sciences. The "marginal concept"—the use of differentials as an approximation principle—is clearly presented as part of the discussion of derivatives, is used again to promote the discussion of optimization problems, and is the basis of many exercises. The theory of several variables is given careful attention. Level curve diagrams are used extensively to promote discussions of partial derivatives, optimization, boundary extrema, and constrained optimization (Lagrange multipliers). For students in the biological and social sciences, growth and decay models are introduced early (in Chapter 1) and then reappear in discussions of both exponential functions and differential equations. There is also a brief, self-contained introduction to probability density functions in the chapter on integration.

•**Level of rigor** *The concept of limit should be developed in a meaningful context, rather than as a mathematical abstraction.* Derivatives are discussed immediately after the review material in Chapter 1, and the necessary notions of limit are developed intuitively. Average rates of change are introduced first, in various applied situations; the derivative is then presented as the limit value of such averages—as an instantaneous rate of change—accompanied by numerical evidence and applied interpretations. Once limits have become familiar, their formal properties are summarized in a final optional section. Integrals are developed in terms of antiderivatives. Their more complicated interpretation as limits of Riemann sums is placed in a single optional section.

•**Review of prerequisites** *The basic prerequisites are reviewed in Chapter 1 and are keyed to the applications that follow.* Chapter 1 may be read by students on their own. Examples are used throughout to show how this material will be involved in future applications. Section 1.3 introduces most of the terminology used in applications and should be read before students go on to the study of derivatives. An appendix reviews the elementary topics of real numbers, inequalities, and exponent laws.

• **Calculators** *Students should be encouraged to use calculators in and out of class.* Many examples and exercises involve realistic numbers, for which the use of a calculator is natural. We also exploit the availability of calculators in dealing with exponentials and logarithms. A full theory is given in Chapter 4, but there is currently no obstacle to introducing these useful functions early and informally, so students can get used to calculations in which they appear.

• **Course duration** *This textbook has been designed for a one-semester or a two-quarter course.* However, we include many optional topics—more than could be covered in one semester. Certain sections in Chapters 1-3 form the core of the book; the remaining sections may be added according to the needs of the instructor. The later chapters on exponentials and logarithms (Chapter 4), integrals (Chapters 5-6), and several variables (Chapter 7) are completely independent; these topics may be covered in any order or even omitted. In particular, the theory of several variables may be taken up early if the audience is heavily oriented toward the management sciences. The important sections in these chapters are listed below.

The core sections in Chapters 1-3 are:

Chapter 1: Sections 1-3, 5. (Students may read this chapter on their own.)
Chapter 2: Sections 1-6.
Chapter 3: Sections 1-8.

The important sections in the remaining chapters are:

Chapter 4: Sections 1, 2, 4.
Chapter 5: Sections 1-3, 5.
Chapter 6: This chapter is optional.
Chapter 7: Sections 1-5, 7.

We would like to express our appreciation to the reviewers whose comments contributed so much to the text. These include the reviewers for the Second Edition: Richard A. Brualdi (University of Wisconsin, Madison), Daniel S. Drucker (Wayne State University), Richard Semmler (Northern Virginia Community College), and Bruce H. Edwards (University of Florida, Gainesville); as well as the reviewers for the original edition: Frank Warner (University of Pennsylvania), Paul Knopp (University of Houston), Larry Goldstein (University of Maryland), and Richard Joss (California State University, Long Beach).

We would also like to acknowledge our debt to Richard Wallis, our acquisitions editor at Harcourt Brace Jovanovich, for his invaluable advice and guidance during the preparation of the Second Edition, and to the staff at Harcourt Brace Jovanovich for their fine work. We are particularly grateful for the efforts of Marji James as our production editor.

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CONTENTS

CHAPTER 1

Preliminary Concepts

1.1 Functions and Their Graphs

Economics, business administration, biology, chemistry, and physics are frequently concerned with the dependence of one quantity upon others. For example, a manufacturer might want to know how profit varies with production level. Or in a typical biological problem, we might study the dependence of the size of a bacterial population on time, temperature, or the concentration of certain nutrients.

Mathematicians have abstracted from these problems the concept of function. The quantities to be measured and compared are called **variables**. To keep things simple, in this chapter we shall compare just two variables. Functions involving several variables make their appearance in Chapter 7. A **function** is a rule or formula that gives the value of one variable if the value of the other variable is specified. For example, temperatures measured on the Celsius (centigrade) and Fahrenheit scales are connected by a simple formula

$$F = \frac{9}{5}C + 32, \qquad [1]$$

where F is degrees Fahrenheit, and C is degrees Celsius.

If a newspaper reports that the temperature in Paris was 20°C, this means that the Fahrenheit temperature was $(\frac{9}{5} \times 20) + 32 = 68$°F—a pleasant day, rather than one on which we should worry about our antifreeze. Here the variables are F and C; the function is the rule [1] giving the value of F when C is specified.

In a simple manufacturing problem, the variables are the profit P (which might be measured in dollars per month) and the production level x (the number of units produced per month). A factory manager can set production at various levels. Once the production level has been set, the profit is determined. If the manager changes the number of units produced per month, then of course the monthly profits change, too. The variables P and x play rather different roles here. The production level x may be set as the manager wishes, within reasonable limits. This is (unfortunately) not true of the profit variable. Asymmetry in the roles of the variables is typical of most functions. Usually, one variable may be varied freely, while the other is determined by the function. The "free" variable is called the **independent variable** and the other the **dependent variable**. In [1] the independent variable is C, and F depends upon it. In a manufacturing problem the production level x is the independent variable, and the profit P is the dependent variable.

Example 1 Apples are priced at 49¢ per pound. Describe the relationship between the cost and the quantity bought.

Solution The natural independent variable x is the number of pounds bought. The dependent variable then, is the price paid, which we denote by y (measured in cents). The function relating y to x is concisely expressed by the formula

$$y = 49x \qquad \text{for } x \geq 0.$$

Notice that this relationship has meaning in this situation only for $x \geq 0$. ■

There is a standard notation for functions. If we let x stand for the independent variable and y for the dependent variable, and if f is some function that relates x to y, the symbol

$$f(x) \quad \text{(which we read "}f\text{ of }x\text{")}$$

stands for the value of the dependent variable y when the independent variable is equal to x. This notation saves a lot of verbiage. Thus if $y = 4$ when $x = -3$, we can just write $4 = f(-3)$. If we are told that $1 = f(2)$, then y has the value 1 when $x = 2$.

Many functions are given by an algebraic formula. For example:

$$f(x) = x^2 - x + 1$$

$$f(x) = \frac{9}{5}x + 32$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{1}{1 + x^2}$$

$$f(x) = 1 \qquad \text{(constant function; } y = 1 \text{ for all } x\text{)}$$

Example 2 If $f(x) = x^2 - x + 1$, what is the value of this function when $x = 2$? When $x = -1$? When $x = 0$? When $x = \frac{1}{2}$?

Solution We find the value by inserting the designated x value into the formula. Thus,

$$f(2) = (2)^2 - 2 + 1 = 4 - 2 + 1 = 3$$

$$f(-1) = (-1)^2 - (-1) + 1 = 1 + 1 + 1 = 3$$

$$f(0) = (0)^2 - (0) + 1 = 1$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 1 = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$$ ■

Example 3 Pure antifreeze (ethylene glycol) freezes at -11.5°C and boils at 198°C. Use the conversion formula [1]

$$F(C) = \frac{9}{5}C + 32$$

to determine the corresponding freezing and boiling points on the Fahrenheit scale.

Solution Setting $C = -11.5$ in this function, we get the freezing point

$$F(-11.5) = \frac{9}{5}(-11.5) + 32 = 11.3°\text{F}$$

Similarly, the boiling point is

$$F(198) = \frac{9}{5}(198) + 32 = 388.4°\text{F}$$ ■

Although mathematicians often let x stand for the independent variable and y for the dependent variable, they occasionally use more suggestive symbols, such as C and F in Example 3. After all, the names we give the variables do not really matter. What counts is the way one variable depends on the other.

Sometimes the independent variable can take on only restricted values. In Example 1, the cost of apples makes little sense for negative x, so attention is restricted to $x \geq 0$. In manufacturing problems, negative levels of production would not have much meaning. The set of "allowable" or "feasible" values for the independent variable is called the **feasible set** or **domain of definition** of the function.† In Example 1 the domain consists of all numbers $x \geq 0$.

Example 4 A study conducted by the XYZ Manufacturing Company shows that their monthly profit P depends on the number x of units produced per month according to the formula

$$P(x) = -3800 + 15x - 0.005x^2$$

† Mathematicians favor "domain of definition" or simply "domain," but "feasible set" is commonly used and sounds more natural in economics. We will use both terms interchangeably so the reader will feel at home with either.

Operating full time, the plant can produce at most 1000 units per month. What is the feasible set of production levels? Find P when $x = 300$. When $x = 0$. When $x = 1000$.

Solution Feasible production levels must satisfy the conditions

(i) $x \geq 0$ (negative production levels make no sense)
(ii) $x \leq 1000$ (full-time plant capacity is 1000 units per month)

so the feasible set is the interval $0 \leq x \leq 1000$. When $x = 300$ the profit is

$$\begin{aligned} P = P(300) &= -3800 + 15(300) - 0.005(300)^2 \\ &= -3800 + 4500 - 450 \\ &= \$250 \end{aligned}$$

Similarly,

$$P(0) = -3800 + 15(0) - 0.005(0)^2 = \$-3800$$

(negative profit means they are losing money for the month), and

$$\begin{aligned} P(1000) &= -3800 + 15(1000) - 0.005(1000)^2 \\ &= -3800 + 15{,}000 - 5000 \\ &= \$6200 \end{aligned}$$

■

If $f(x)$ is given by a formula without any mention of a domain of definition, the domain is taken to be all x where the formula makes sense. Thus, if $f(x) = x^2 - x + 1$, the domain is all x. But if

$$f(x) = \frac{1}{x}$$

this makes sense only if $x \neq 0$. So the domain is all $x \neq 0$. Similarly, if

$$f(x) = \sqrt{x}$$

this makes sense only if $x \geq 0$.

Not all functions are given by algebraic formulas. Here is one example.

Example 5 U.S. postage for a first-class letter depends on its weight w. The cost c is 20¢ for the first ounce (or less) and 17¢ for each additional ounce or fraction thereof. Describe the function $c(w)$. Find the cost of a 3.5-ounce letter.

Solution The weight w (in ounces) is the independent variable. It must be positive, so the feasible set is all $w > 0$. For small letters, say under 4 ounces, the function may be described by a table of values

$$c = \begin{cases} 20 & \text{if } 0 < w \leq 1 \\ 37 & \text{if } 1 < w \leq 2 \\ 54 & \text{if } 2 < w \leq 3 \\ 71 & \text{if } 3 < w \leq 4 \end{cases}$$

From this we see that $c(3.5) = 71$¢. ■

It may be an overstatement to say that "a picture is worth a thousand words," yet a picture of a function is very useful. This picture is called the graph of the function. In a plane, let us draw horizontal and vertical coordinate axes so that these perpendicular axes meet at the point corresponding to zero on each line (see Figure 1.1). The point where they meet is called the **origin**. In this **coordinate plane** we may label each point by a pair of numbers (a, b) called the **coordinates** of the point. These coordinates tell us how to find the point: The first number a tells us to move a units horizontally, starting from the origin. The second number b tells us then to move b units parallel to the vertical axis. If either a or b is negative, we must move parallel to the appropriate axis—but in the *negative* direction. For example, the origin O has coordinates $(0, 0)$. To locate the point Q with coordinates $(1, -2)$ as shown in Figure 1.1, we move 1 unit to the right along the horizontal axis, and then -2 units (2 units *down*) parallel to the vertical axis.

The **graph** of a function f is the set of points (x, y) in the plane such that $y = f(x)$. The general idea is shown in Figure 1.2, where we indicate the graph points corresponding to $x = -1, 0, 1, 2$. There is one point on the graph above

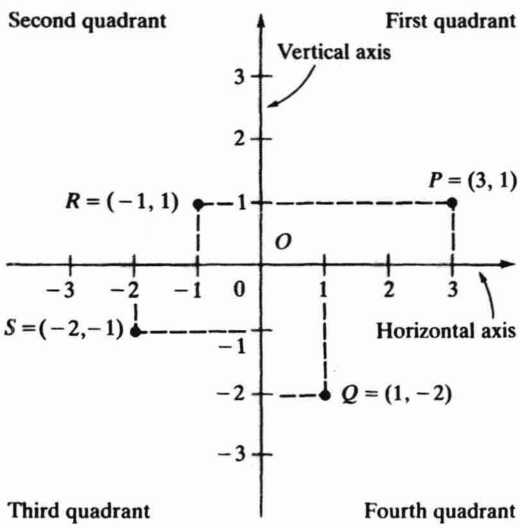


Figure 1.1
The point P in the coordinate plane has coordinates $(3, 1)$ because we reach it by moving $+3$ units parallel to the horizontal axis and $+1$ units parallel to the vertical axis. The origin O has coordinates $(0, 0)$. Coordinates of other points Q, R, S are indicated. The coordinate axes divide the plane into four pieces called *quadrants*, which are labeled as shown.

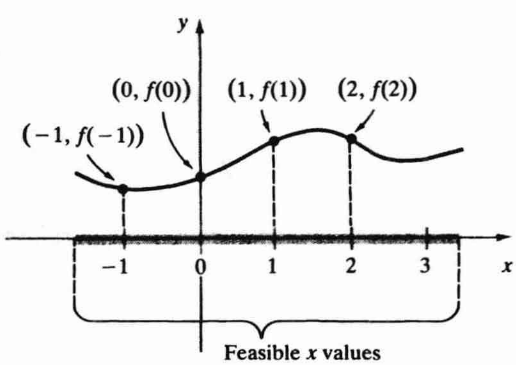
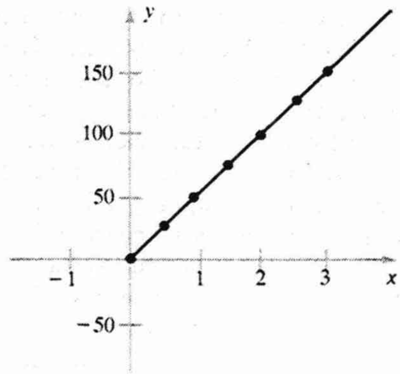


Figure 1.2
The graph of $f(x)$ consists of all points of the form $(x, y) = (x, f(x))$, where x is in the feasible set (indicated by the shaded interval on the x axis). Graph points for $x = -1, 0, 1, 2$ are shown as solid dots.



x	$y = 49x$
0	0.0
$\frac{1}{2}$	24.5
1	49.0
$\frac{3}{2}$	73.5
2	98.0
$\frac{5}{2}$	122.5
3	147.0

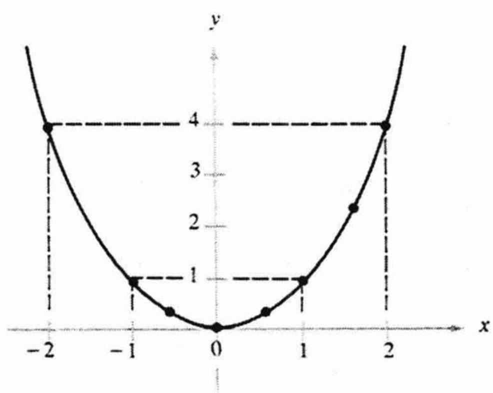
Figure 1.3
The function $y = 49x$ is defined for $x \geq 0$. For selected values of x the values of $y = 49x$ are tabulated at the right. The line indicates the location of *all* points on the graph. For convenience we use different scales of length on the two coordinate axes. The unit length on the vertical axis is much smaller than the unit length on the horizontal axis. This strategy allows us to sketch the graph in a reasonable amount of space.

(or below) each x in the domain of definition; there are *no* graph points corresponding to other x values. As an example, let us draw the graph of the function $f(x) = 49x$ (for $x \geq 0$) discussed in Example 1. We do this by plotting enough points on the graph to suggest its shape; then we draw a smooth curve through these points. In Figure 1.3 we have listed several allowable values of x and have computed the corresponding values of $y = 49x$. Each entry in the table gives the coordinates $(x, y) = (x, 49x)$ of a point on the graph. For example, $y = 98$ if $x = 2$, so $(2, 98)$ is on the graph. These graph points are indicated by solid dots in the figure. The complete graph is part of a straight line beginning at the origin. Because the function is not defined for $x < 0$, there are no graph points lying above or below negative values of x on the horizontal axis.

In sketching graphs we always associate the independent variable with the horizontal axis and the dependent variable with the vertical axis, as in Figure 1.3.

Example 6 Sketch the graph of the function $f(x) = x^2$.

Solution First plot the graph points for a few values of x. The solid dots in Figure 1.4 correspond to the tabulated values of x and $y = x^2$. A few additional remarks will help us make an accurate sketch.



x	$y = x^2$
−2	4.0
−1	1.0
−0.5	0.25
0	0.0
0.5	0.25
1	1.0
1.5	2.25
2	4.0

Figure 1.4
Graph of $y = x^2$. Points on the graph above $+x$ and $-x$ have the same height above the x axis, namely $y = x^2$. This is shown for $x = \pm 1$ (which correspond to $y = 1$) and $x = \pm 2$ (which correspond to $y = 4$). Thus the graph is symmetric about the vertical axis.

The square $y = x^2$ of any number x is positive, whether x is positive or negative. Thus the points on the graph all lie in the upper half of the coordinate plane. And $y = f(x)$ assumes the same value at $+x$ and $-x$, so that the points on the graph above $+x$ and $-x$ have the same height above the x axis, namely $y = x^2 = (-x)^2$. This means that the graph is symmetric with respect to the y axis, as shown in Figure 1.4. Finally, as x moves right along the x axis, taking on large positive values (or to the left, taking on large negative values), the dependent variable $y = x^2$ assumes large positive values. With these general observations in mind we can sketch the curve shown in Figure 1.4. ■

Example 7 Draw the graph of the postage function in Example 5, relating c = cost to w = weight.

Solution Here the variables are labeled w and c instead of x and y. If w lies in the range $0 < w \leq 1$, the related value of c is 20. Thus, as w moves from 0 to 1 along the horizontal axis, the corresponding points (w, c) on the graph all have the same height $c = 20$ above the horizontal axis; they lie on a horizontal line segment 20 units above the horizontal axis. If $1 < w \leq 2$, then $c = 37$; so the corresponding points on the graph lie on a horizontal line segment 37 units above the w axis. A similar analysis shows that the graph consists of horizontal segments, as shown in Figure 1.5. ■

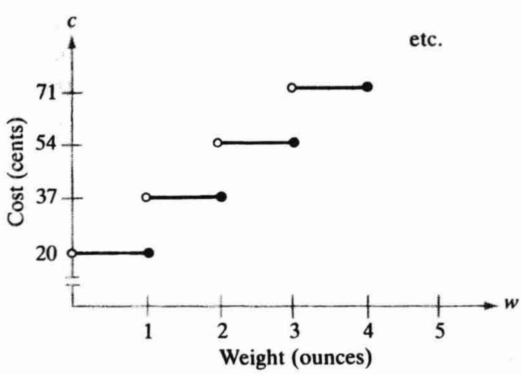


Figure 1.5
The graph of the postage function $c = c(w)$ in Example 7 is a "step function." An open dot indicates that the point is *not* included in the graph; solid dots *are* on the graph. Note the sudden breaks in the graph at $w = 1$, $w = 2$, and $w = 3$. Compare this with the smoothness of the graph in Figure 1.4. There is no algebraic formula for the postage function, though there is a perfectly well-defined logical rule for calculating $c(w)$.

General observations about the behavior of $f(x)$ as x moves far to the left or right can be very useful in making an accurate sketch from a small number of plotted points. In the next example, we shall use a simple fact about reciprocals:

$$\frac{1}{\text{(small positive)}} = \text{(large positive)}$$
$$\frac{1}{\text{(large positive)}} = \text{(small positive)} \qquad [2]$$

For example, 200,000 is quite a larger number, but $1/200{,}000 = 0.000005$ is very small. For negative values, the outcome, determined by the rule of signs $(+)/(-) = (-)$, is similar.

Example 8 Sketch the graph of the function $y = 1/x$.

Solution The function is not defined at $x = 0$; there is no point on the graph above or below $x = 0$. Generally, a graph has some sort of singular behavior wherever an algebraic formula for the function has denominator zero. It is a good idea to plot several points near such values of x (here $x = 0$). The general principle [2] can also be helpful. In Figure 1.6 we have tabulated some values of

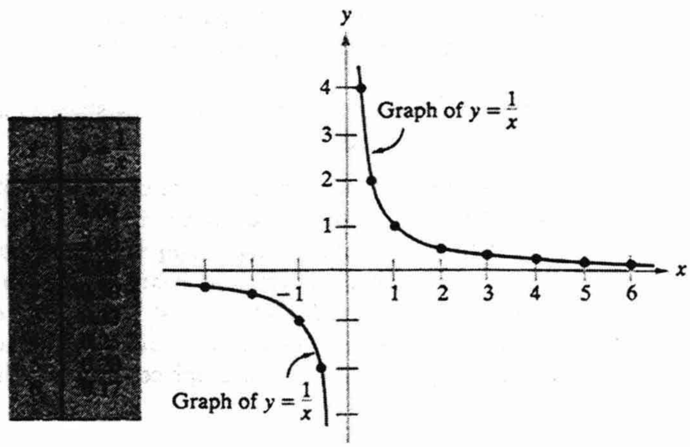


Figure 1.6
The graph of $y = 1/x$, defined for $x \neq 0$, consists of two separate curves. Values of $y = 1/x$ are tabulated for selected positive x; the value at $-x$ is (-1) times the value at $+x$, so there is no need to make a separate table for negative x.

$y = 1/x$, taking several values of x near the troublesome point $x = 0$. We can save ourselves the trouble of calculating $f(x)$ for negative values of x by noticing that $f(-x) = (-1)\cdot f(x)$; the value at $-x$ is the negative of the value at x. Let us see what happens for small positive or small negative values of x.

$$x \text{ small positive; } y = \frac{1}{x} \text{ large positive} \quad \text{(graph point high above } x \text{ axis)}$$

$$x \text{ small negative; } y = \frac{1}{x} \text{ large negative} \quad \text{(graph point far below } x \text{ axis)}$$

This behavior is shown in the final sketch (Figure 1.6). What happens as x moves far to the right? The corresponding graph point approaches the x axis, but is slightly above it because $y = 1/(\text{large positive}) = (\text{small positive})$. ■

Exercises 1.1

1. If $f(x) = 3x - 2$, find the values $f(0)$, $f(-1)$, $f(1)$, and $f(8)$. For what value of x is $f(x) = 4$?

2. If $f(x) = x^2 - 2$, what is the value of $y = f(x)$ when
(i) $x = -1$ (ii) $x = 2$ (iii) $x = 0$ (iv) $x = \sqrt{2}$

Find the values $f(-2)$, $f(0)$, $f(\frac{1}{2})$, $f(1)$, and $f(5)$ for the following functions.

3. $f(x) = 4 - 13x$

4. $f(x) = 5 - x + 2x^2$

5. $f(x) = 1 + x - x^2$

6. $f(x) = x^3 - 3x + 2$