DOVER PHOENIX EDITIONS

Topology

An Introduction with Application to Topological Groups

George McCarty

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Preface

THIS TEXT introduces the student to that part of geometry which is generally labeled "topology." It will give him that familiarity with elementary point set topology, including an easy acquaintance with the line and the plane, which has become prerequisite to most graduate programs in mathematics. Nevertheless, it is not a collection of such topics; rather, it early employs the language of point set topology to define and discuss topological groups. These geometric objects in turn motivate a further discussion of set-theoretic topology and of its applications in function spaces. An introduction to homotopy and the fundamental group then brings the student's new theoretical knowledge to bear on very concrete problems: the calculation of the fundamental group of the circle and a proof of the fundamental theorem of algebra. Finally, the abstract development is brought to a satisfying fruition with the classification of topological groups by equivalence under local isomorphism.

There is general agreement that every serious student of mathematics should take part in some sustained, deep geometric development. This text is quite close, for instance, to the recommendations of CUPM for a one-year course on set-theoretic topology.* Such a sustained development might be considered complete, for some students, at the end of Chapter IX, with the study of local isomorphism classes omitted. However, much significance will be lost to the student who fails to reach the fundamental

* These recommendations are outlined on pages 61–64 of the pamphlet *Preliminary Recommendations for Pregraduate Preparation of Research Mathematicians*, published by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America, May, 1963.

theorem of algebra (that is, to finish Chapter IX). Thus an instructor may need considerable flexibility of material to adjust the global schedule of his course. Many extra topics are provided, any of which may be omitted without disturbing the continuity of reasoning; these are gathered at the very end of each chapter as PROBLEMS. Much of this material may be discussed in class during a one-year course, or it might be omitted entirely for a shorter presentation. The easy questions which every student should consider are labeled EXERCISES and grouped in front of the PROBLEMS. Some, but not all, of the EXERCISES are cited at the ends of those sections where they may first be conveniently assigned.

Linear algebra is drawn upon for many examples in later chapters; although this material could be deleted, an introduction to vectors and matrices will normally be prerequisite to the course. With the thought that some excellent young students might wish to enroll in the prerequisite course concurrently, I have made no mention of linear algebra prior to Chapter VII. However, an instructor can easily add linear examples and exercises to the chapters on groups and on topological groups. An understanding of mathematical induction and of the completeness of the reals is assumed, with the first usage of each concept signaled by footnote. A knowledge of modern algebra is not presumed; of course, a class trained in elementary group theory could regard Chapter II as a review (although its emphasis on infinite groups may be news). At the other extreme, this text may be used as a short course on topological groups for a class already familiar with point set topology; parts of Chapters V and VII through X will be pertinent.

A feature of the text is its emphasis on the quotient-function–equivalence concept; a uniform treatment in the contexts of sets, groups, spaces, and then topological groups stresses aspects common to all these settings. Although functors and categories are nowhere defined, at each appearance of a new functor its algebraic properties are derived and emphasized. Additionally, several problems explore categorical characterizations of various concepts.

There are a few didactic novelties here. A "Cayley theorem" is offered for topological groups. The triangle law for a product metric is proved in a way new to me. The fact that locally isomorphic groups have globally isomorphic based-path groups is made the basis of the classification theorem. And a canonical factorization of functions (and morphisms) results in the composite of a quotient function, a 1-1 correspondence, and an inclusion function.

As did many topologists, I cut my teeth on Pontrjagin's *Topological Groups*. Among my numerous other debts are those to Richard Arens,

Saunders Mac Lane, and C. B. Tompkins; each of these men has, by his unique example, taught me to think.

Students at Harvey Mudd College and at the University of California at Irvine who studied early versions have, by their kind suggestions, improved this book in many ways. Edwin Spanier has helped me with several comments on mathematical exposition. The entire manuscript was read by Robert F. Brown; without his many wise criticisms and his corrections of errors of mathematics, exposition, style, and English, this book would not be complete.

George McCarty

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Introduction

This book is about topology. You may have read popular articles about "rubber-sheet geometry"; at this point it is difficult to say more precisely just what is studied in this branch of mathematics. We shall examine many mathematical objects, both familiar and strange: the real, complex, and quaternionic number systems, universal covering spaces and fundamental groups, spheres and projective spaces, deck transformations, and the compact-open topology (to name a few you might already be curious about). At the end, you will be able to form your own partial answer to the question "what is topology?"

But between here and the end lies hard work for you, as well as fun. Topology is a part of mathematics; this theory is applied in many fields, from quantum mechanics to sociology, and we shall point out some of these applications. But we must first build up the vocabulary of a theoretical structure. We shall construct our theory abstractly, with axioms, and our major effort will be the exploration of the theoretical consequences of those axioms. This contrasts with the calculus, where usually one strives mainly to acquire a competence in solving specific problems; a theorem is often regarded as a recipe for applying the formulas to special cases. Here you will strive to understand the theorems so thoroughly that you yourself can invent proofs for new theorems. In short, really study the proofs!

EXERCISES AND PROBLEMS

You will get exercise, sometimes vigorous exercise, as you read this book. It requires work to fill in the detailed reasoning from one sentence to the next; when the jump is particularly large we shall sometimes signal this with such euphemisms as "it is easy to see that . . ." and "it is obvious (or clear) that . . ."; other times the word "why?" may be interjected in parentheses to remind you that a few details are missing. Fill in these details mentally as you read; you will find that you become better and better at this with practice.

However, you want to learn to use mathematics, not just to read it. And you cannot learn that by osmosis, by watching someone else do it, any more than you could learn to play chess or football by close observation. There are EXERCISES at the end of each chapter; *do them all*. They are not repetitive drills; you can expect some of the joy of discovery and creation with each solution you construct. Furthermore, you will learn topology, chapter by chapter, through your exercises; we shall count on your efforts by presuming, as each new topic is presented, that you understand the results of those exercises that have appeared earlier. With experience, you will know which you can do in your head and which are difficult for you; write out detailed solutions for the harder ones.

In each chapter, following the exercises, there is a set of PROBLEMS. You will need neither the results of the problems nor the exercise of working them to continue your reading of the book. Instead, they contain interesting applications and further theory, a sort of payoff for your work in the chapter.

INTERNAL REFERENCES

Throughout the book we have tried to minimize the number of formal references to previous material, preferring instead to refer to a theorem by name or a brief statement of the result. However, we have found it necessary to number some of the statements in Chaps. I, II, and IV. A reference in Chap. II, for instance, to Theorem 3 means Theorem 3 of that chapter, while a reference there to Theorem I.3 means Theorem 3 of Chap. I. Similarly, a reference in Chap. II to Exercise A means Exercise A of Chap. II, while reference there to Exercise I.B means the second exercise of Chap. I. Problems are labeled with doubled letters to distinguish them from exercises; a reference to Prob. CC is to the third problem at the end of the chapter where the reference appears.

Should you not recall the content of a theorem, such as the Quotient Theorem for Groups, when it is referred to, you will find it listed in the index.

DEFINITIONS

We shall define new words in two ways: A defined word in boldface (a **thingamabob** is an orange whatsit) is part of our minimal vocabulary; it will be used later and *must be memorized.* A word appearing in quotation marks in its definition (a set of three skeletons is called a "full closet") will not be required later in this book. It is provided for your reference use; it may appear in a parallel text you use, or may be preferred by your instructor. Quotation marks are also used to set off "suggestive" statements which are not part of a formal argument; this usage will be self-explanatory.

SET-THEORETIC NOTATION

The concept of a **set** and operations involving sets, along with logical arguments expressed in ordinary English, form the language of this book. A set (synonyms: **class, family, collection**) is, intuitively, a bunch, aggregate, flock, etc.† All statements about sets are to be made within some "large" set called the **universe,** which contains every set in view. The particular choice of universe may change from topic to topic, but either it will always be clearly understood or it will be made explicit just which universe is being used in the discussion at hand. For example, if S is defined to be the set of all positive real numbers in a context where the universe is understood to be the set **R** of all real numbers, then the complement of S is the set of real numbers which are either negative or zero. But if, in a different context, the real numbers are themselves considered to be a subset of the universe **C** of all complex numbers (that is, **R** is the x-axis), then the complement of S in **C** is the set $\{x + iy\text{: either } y \neq 0 \text{ or } x \leq 0\}$, that is, the whole plane except for S.

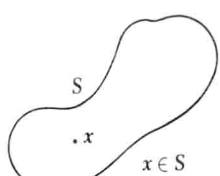
The "set-braces" notation of the sentence above will be used frequently.

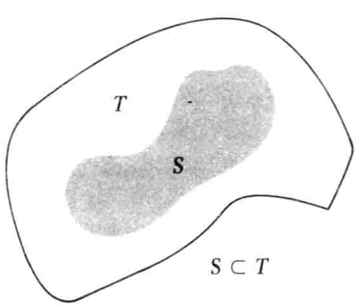
† A pod (of whales) or an exaltation (of larks)!

That is, let $P(x)$ be a statement involving the variable x such that for each particular value a of the variable x either $P(a)$ is true or $P(a)$ is false. Then $\{x: P(x)\}$ denotes the set of all those elements a of the universe for which $P(a)$ is true. The braces are also used to contain implicit or explicit lists of the elements of a set. Self-explanatory examples of this in the universe **R** of real numbers are $R = \{1,3,10\}$, $S = \{1, 2, 3, \ldots, 27\}$, and $T = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$.

We now offer a condensed description of the elementary set-theoretic operations and their nomenclature. This is provided as a rapid review and reference list. If you are not quite familar with some of the ideas involved, consult the references listed at the end of Chap. I (on page 24) for more leisurely introductions.

The membership of an object x in a set S is denoted by $x \in S$. While the meaning of this symbol is fixed, you may read it (and many other mathematical symbols) in various ways in English sentences, depending on context; examples are "... x, which is a member of S, ...," "... x is a member of S ...," "... (let) x be a member of S ...," etc. If $x \in S$, then x is a **member, element,** or **point** of S; because of this nomenclature, sets are sometimes referred to as "point sets." If every member of a set S is also a member of





the set T, then S is said to be **contained** (or **included**) in T, written $S \subset T$, or T **contains** (or **includes**) S, written $T \supset S$; S is then a **subset** (or **subfamily**) of T and T is a "superset" of S. Two sets S and T are defined to be **equal,** written $S = T$, if both $S \subset T$ and $T \subset S$. This *definition* of equality of sets seems to subvert the usual convention that "$A = B$" means that "A" and "B" are two names for the same object. If you cleave to that meaning of equality, this definition of equality of sets may be understood, in our intuitive set theory, to say exactly what a set is: the totality of its members and nothing else.

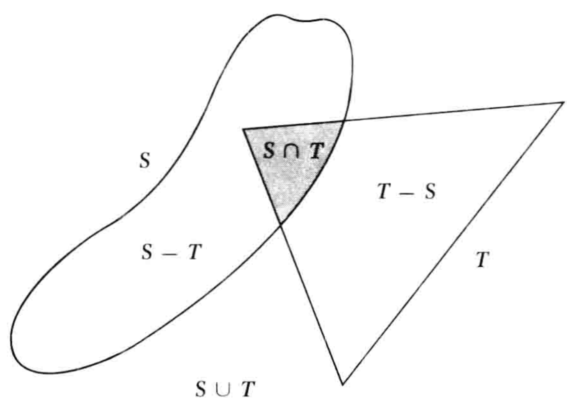
Each of the above symbols may be negated by the addition of a slanted stroke: $\notin$, $\not\subset$, $\not\supset$, $\neq$. The definitions of the resulting symbols are clear; for instance, $S \neq T$ means that either $S \not\subset T$ or $T \not\subset S$. (In this last sentence and throughout mathematics, the word "or" is used in the nonexclusive sense; the words "or both" are understood.) If $S \neq T$ and $S \subset T$, then S is a **proper subset of** T. The **complement of** S **in** T, $T - S$ (read "T minus S"), is the set of elements of T which are not elements of S,

$$T - S = \{x: x \in T \text{ and } x \notin S\}.$$

The **complement** S' of a set S is the set of nonelements of S; if U denotes the universe, then $S' = U - S$.

The **intersection** of two sets S and T, $S \cap T$ (read "S intersect T"), is the set of all elements common to both S and T,

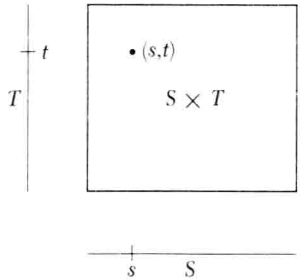
$$S \cap T = \{x: x \in S \text{ and } x \in T\}.$$



The **union** of S and T, $S \cup T$ (read "S union T"), is the set of all elements which belong to either S or T,

$$S \cup T = \{x: x \in S \text{ or } x \in T\}.$$

For any objects a and b, let (a,b) denote the **ordered pair,** with a first and b second. Two ordered pairs (a,b) and (a',b') are equal if and only if both $a = a'$ and $b = b'$. The **direct** (or "cartesian") **product** of S and T is the set $S \times T$ of all ordered pairs whose whose first element is a member of S and whose second element is a member of T,



$$S \times T = \{(s,t): s \in S \text{ and } t \in T\}.$$

The sets S and T are called the **factors** of $S \times T$.

LOGIC

The word **implies** is used in this book in its mathematical sense: it is *not* intended to have its English-language meaning. Rather, if we say "P implies Q," where P is a mathematical statement which might be either true or false, and Q is another such statement, we mean "if P is true, then Q is true." The statement "P implies Q" is true unless both P is true and Q is false; that is, "P implies Q" is true if either Q is true or P is false. This precise definition results in some true statements being outrageous when read in the common language; "$1 = 0$ implies $1 = 1$" is an instance, as is "$1 = 0$ implies $1 = 3$." Notice that the statements "P implies Q," "if Q is false, then P is false," and "P is true only if Q is true" are logically equivalent.

SPECIAL SYMBOLS

We shall frequently use the symbol **iff** to stand for the words "if and only if"; thus "P iff Q" means both P implies Q and Q implies P.

The shorthand symbol ■ has become more fashionable than Q.E.D. as a signal to the reader that a proof is concluded. We shall use it for that; however, its appearance immediately following the statement of a theorem will indicate either that the proof preceded the statement or that the proof is omitted. Whenever a proof is omitted, it is easy; construct it in your head at once.

A few particular sets will be discussed enough to make special symbols helpful for their recognition. Some of these are

- $\varnothing$ the **empty** ("void," "null") set, $\varnothing = \{x: x \neq x\}$,
- $\mathbf{Z}$ the set of all integers, $\mathbf{Z} = \{\ldots, -1, 0, 1, 2, 3, \ldots\}$,
- $\mathbf{R}$ the set of all real numbers,
- $\mathbf{C}$ the set of all complex numbers,
- $\mathbf{I}$ the closed interval of real numbers between 0 and 1, $\mathbf{I} = [0,1] = \{x: x \in \mathbf{R} \text{ and } 0 \leq x \leq 1\}$.

A complete index of special symbols is given immediately in front of the index, along with a Greek alphabet, for your reference.