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A HISTORY OF THE MATHEMATICAL THEORY OF PROBABILITY

FROM THE TIME OF PASCAL TO THAT OF LAPLACE

ISAAC TODHUNTER

For with the notation of Art. 957 we have $y = x^m (1-x)^n$; the value of x which makes y a maximum is found from the equation

$$\frac{m}{x} - \frac{n}{1-x} = 0,$$

so that
$$a = \frac{m}{m+n}$$

Then

$$t^2 = \log \frac{Y}{(a+\theta)^m (1-a-\theta)^n}$$

$$= \log \frac{Y}{a^m (1-a)^n} - m \log\left(1+\frac{\theta}{a}\right) - n \log\left(1 - \frac{\theta}{1-a}\right)$$

$$= \frac{\theta^2}{2}\left\{\frac{m}{a^2} + \frac{n}{(1-a)^2}\right\} - \frac{\theta^3}{3}\left\{\frac{m}{a^3} - \frac{n}{(1-a)^3}\right\} + \dots$$

Thus, approximately,

$$t^2 = \frac{\theta^2}{2}\left\{\frac{m}{a^2} + \frac{n}{(1-a)^2}\right\} = \frac{\theta^2 (m+n)^3}{2mn}.$$

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A History of the Mathematical Theory of Probability

Throughout his early life, Isaac Todhunter (1820–84) excelled as a student of mathematics, gaining a scholarship at the University of London and numerous awards during his time at St John's College, Cambridge. Taking up fellowship of the college in 1849, he became widely known for both his educational texts and his historical accounts of various branches of mathematics. The present work, first published in 1865, describes the rise of probability theory as a recognised subject, beginning with a discussion of the famous 'problem of points', as considered by the likes of the Chevalier de Méré, Blaise Pascal and Pierre de Fermat during the latter half of the seventeenth century. Subsequently, the application of advanced methods that had been developed in classical areas of mathematics led to rapid progress in probability theory. Todhunter traces this growth, closing with a thorough account of Pierre-Simon Laplace's far-reaching work in the area.

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HISTORY OF

THE THEORY OF PROBABILITY.

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FROM THE TIME OF PASCAL TO THAT OF LAPLACE.

BY

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PREFACE.

THE favourable reception which has been granted to my *History of the Calculus of Variations during the Nineteenth Century* has encouraged me to undertake another work of the same kind. The subject to which I now invite attention has high claims to consideration on account of the subtle problems which it involves, the valuable contributions to analysis which it has produced, its important practical applications, and the eminence of those who have cultivated it.

The nature of the problems which the Theory of Probability contemplates, and the influence which this Theory has exercised on the progress of mathematical science and also on the concerns of practical life, cannot be discussed within the limits of a Preface; we may however claim for our subject all the interest which illustrious names can confer, by the simple statement that nearly every great mathematician within the range of a century and a half will come before us in the course of the history. To mention only the most distinguished in this distinguished roll—we shall find here—Pascal and Fermat, worthy to be associated by kindred genius and character—De Moivre with his rare powers of analysis, which seem to belong only to a later epoch, and which justify the honour in which he was held by Newton—Leibnitz and the eminent school of which he may be considered the founder, a school including the Bernoullis and Euler—D'Alembert, one of the most conspicuous of those who brought on the French revolution, and Condorcet, one of the most illustrious of its victims—Lagrange and Laplace who survived until the present century, and may be regarded as rivals at that time for the supremacy of the mathematical world.

I will now give an outline of the contents of the book.

The first Chapter contains an account of some anticipations of the subject which are contained in the writings of Cardan, Kepler and Galileo.

The second Chapter introduces the Chevalier de Méré who having puzzled himself in vain over a problem in chances, fortunately turned for help to Pascal: the Problem of Points is discussed in the correspondence between Pascal and Fermat, and thus the Theory of Probability begins its career.

The third Chapter analyses the treatise in which Huygens in 1659 exhibited what was then known of the subject. Works such as this, which present to students the opportunity of becoming acquainted with the speculations of the foremost men of the time, cannot be too highly commended; in this respect our subject has been fortunate, for the example which was afforded by Huygens has been imitated by James Bernoulli, De Moivre and Laplace—and the same course might with great advantage be pursued in connexion with other subjects by mathematicians in the present day.

The fourth Chapter contains a sketch of the early history of the theory of Permutations and Combinations; and the fifth Chapter a sketch of the early history of the researches on Mortality and Life Insurance. Neither of these Chapters claims to be exhaustive; but they contain so much as may suffice to trace the connexion of the branches to which they relate with the main subject of our history.

The sixth Chapter gives an account of some miscellaneous investigations between the years 1670 and 1700. Our attention is directed in succession to Caramuel, Sauveur, James Bernoulli, Leibnitz, a translator of Huygens's treatise whom I take to be Arbuthnot, Roberts, and Craig—the last of whom is notorious for an absurd abuse of mathematics in connexion with the probability of testimony.

The seventh Chapter analyses the *Ars Conjectandi* of James Bernoulli. This is an elaborate treatise by one of the greatest mathematicians of the age, and although it was unfortunately left incomplete, it affords abundant evidence of its author's ability and of his interest in the subject. Especially we may notice the famous theorem which justly bears the name of James Bernoulli, and which places the Theory of Probability in a more commanding position than it had hitherto occupied.

The eighth Chapter is devoted to Montmort. He is not to be compared for mathematical power with James Bernoulli or De Moivre; nor does he seem to have formed a very exalted idea of the true dignity and importance of the subject. But he was enthusiastically devoted to it; he spared no labour himself, and his influence direct or indirect stimulated the exertions of Nicolas Bernoulli and of De Moivre.

The ninth Chapter relates to De Moivre, containing a full analysis of his *Doctrine of Chances.* De Moivre brought to bear on the subject mathematical powers of the highest order; these powers are especially manifested in the results which he enunciated respecting the great problem of the Duration of Play. Unfortunately he did not publish demonstrations, and Lagrange himself more than fifty years later found a good exercise for his analytical skill in supplying the investigations; this circumstance compels us to admire De Moivre's powers, and to regret the loss which his concealment of his methods has occasioned to mathematics, or at least to mathematical history.

De Moivre's *Doctrine of Chances* formed a treatise on the subject, full, clear and accurate; and it maintained its place as a standard work, at least in England, almost down to our own day.

The tenth Chapter gives an account of some miscellaneous investigations between the years 1700 and 1750. These investigations are due to Nicolas Bernoulli, Arbuthnot, Browne, Mairan, Nicole, Buffon, Ham, Thomas Simpson and John Bernoulli.

The eleventh Chapter relates to Daniel Bernoulli, containing an account of a series of memoirs published chiefly in the volumes of the Academy of Petersburg; the memoirs are remarkable for boldness and originality, the first of them contains the celebrated theory of Moral Expectation.

The twelfth Chapter relates to Euler; it gives an account of his memoirs, which relate principally to certain games of chance.

The thirteenth Chapter relates to D'Alembert; it gives a full account of the objections which he urged against some of the fundamental principles of the subject, and of his controversy with Daniel Bernoulli on the mathematical investigation of the gain to human life which would arise from the extirpation of one of the most fatal diseases to which the human race is liable.

The fourteenth Chapter relates to Bayes; it explains the method by which he demonstrated his famous theorem, which may be said to have been the origin of that part of the subject which relates to the probabilities of causes as inferred from observed effects.

The fifteenth Chapter is devoted to Lagrange; he contributed to the subject a valuable memoir on the theory of the errors of observations, and demonstrations of the results enunciated by De Moivre respecting the Duration of Play.

The sixteenth Chapter contains notices of miscellaneous investigations between the years 1750 and 1780. This Chapter brings before us Kaestner, Clark, Mallet, John Bernoulli, Beguelin, Michell, Lambert, Buffon, Fuss, and some others. The memoir of Michell is remarkable; it contains the famous argument for the existence of design drawn from the fact of the closeness of certain stars, like the Pleiades.

The seventeenth Chapter relates to Cordorcet, who published a large book and a long memoir upon the Theory of Probability. He chiefly discussed the probability of the correctness of judgments determined by a majority of votes; he has the merit of first submitting this question to mathematical investigation, but his own results are not of great practical importance.

The eighteenth Chapter relates to Trembley. He wrote several memoirs with the main design of establishing by elementary methods results which had been originally obtained by the aid of the higher branches of mathematics; but he does not seem to have been very successful in carrying out his design.

The nineteenth Chapter contains an account of miscellaneous investigations between the years 1780 and 1800. It includes the following names; Borda, Malfatti, Bicquilley, the writers in the mathematical portion of the *Encyclopédie Méthodique*, D'Anieres, Waring, Prevost and Lhuilier, and Young.

The twentieth Chapter is devoted to Laplace; this contains a full account of all his writings on the subject of Probability. First his memoirs in chronological order, are analysed, and then the great work in which he embodied all his own investigations and much derived from other writers. I hope it will be found that all the parts of Laplace's memoirs and work have been carefully and clearly expounded; I would venture to refer for examples to Laplace's method of approximation to integrals, to the Problem of Points, to James Bernoulli's theorem, to the problem taken from Buffon, and above all to the famous method of Least Squares. With respect to the last subject I have availed myself of the guidance of Poisson's luminous analysis, and have given a general investigation, applying to the case of more than one unknown element. I hope I have thus accomplished something towards rendering the theory of this important method accessible to students.

In an Appendix I have noticed some writings which came under my attention during the printing of the work too late to be referred to their proper places.

I have endeavoured to be quite accurate in my statements, and to reproduce the essential elements of the original works which I have analysed. I have however not thought it indispensable to preserve the exact notation in which any investigation was first presented. It did not appear to me of any importance to retain the specific letters for denoting the known and unknown quantities of an algebraical problem which any writer may have chosen to use. Very often the same problem has been discussed by various writers, and in order to compare their methods with any facility it is necessary to use one set of symbols throughout, although each writer may have preferred his peculiar set. In fact by exercising care in the choice of notation I believe that my exposition of contrasted methods has gained much in brevity and clearness without any sacrifice of real fidelity.

I have used no symbols which are not common to all mathematical literature, except $\underline{|n}$ which is an abbreviation for the product 1.2, ...n, frequently but not universally employed: some such symbol is much required, and I do not know of any which is preferable to this, and I have accordingly introduced it in all my publications.

There are three important authors whom I have frequently cited whose works on Probability have passed through more than one edition, Montmort, De Moivre, and Laplace: it may save trouble to a person who may happen to consult the present volume if I here refer to pages 79, 136, and 495 where I have stated which editions I have cited.

Perhaps it may appear that I have allotted too much space to some of the authors whose works I examine, especially the more ancient; but it is difficult to be accurate or interesting if the narrative is confined to a mere catalogue of titles: and as experience shews that mathematical histories are but rarely undertaken, it seems desirable that they should not be executed on a meagre and inadequate scale.

I will here advert to some of my predecessors in this department of mathematical history; and thus it will appear that I have not obtained much assistance from them.

In the third volume of Montucla's *Histoire des Mathematiques* pages 380—426 are devoted to the Theory of Probability and the kindred subjects. I have always cited this volume simply by the name *Montucla,* but it is of course well known that the third and fourth volumes were edited from the author's manuscripts after his death by La Lande. I should be sorry to appear ungrateful to Montucla; his work is indispensable to the student of mathematical history, for whatever may be its defects it remains without any rival. But I have been much disappointed in what he says respecting the Theory of Probability; he is not copious, nor accurate, nor critical. Hallam has characterised him with some severity, by saying in reference to a point of mathematical history, "Montucla is as superficial as usual:" see a note in the second Chapter of the first volume of the *History of the Literature of Europe.*

There are brief outlines of the history involved or formally incorporated in some of the elementary treatises on the Theory of Probability: I need notice only the best, which occurs in the Treatise on Probability published in the Library of Useful Knowledge. This little work is anonymous, but is known to have been written by Lubbock and Drinkwater; the former is now Sir John Lubbock, and the latter changed his name to Drinkwater-Bethune: see Professor De Morgan's *Arithmetical Books...* page 106, a letter by him in the *Assurance Magazine,* Vol. IX. page 238, and another letter by him in the *Times,* Dec. 16, 1862. The treatise is interesting and valuable, but I have not been able to agree uniformly with the historical statements which it makes or implies.

A more ambitious work bears the title *Histoire du Calcul des Probabilités depuis ses origines jusqu'à nos jours par Charles Gouraud...* Paris, 1848. This consists of 148 widely printed octavo pages; it is a popular narrative entirely free from mathematical symbols, containing however some important specific references. Exact truth occasionally suffers for the sake of a rhetorical style unsuitable alike to history and to science; nevertheless the general reader will be gratified by a lively and vigorous exhibition of the whole course of the subject. M. Gouraud recognises the value of the purely mathematical part of the Theory of Probability, but will not allow the soundness of the applications which have been made of these mathematical formulæ to questions involving moral or political considerations. His history seems to be a portion of a very extensive essay in three folio volumes containing 1929 pages written when he was very young in competition for a prize proposed by the French Academy on a subject entitled *Théorie de la Certitude*; see the *Rapport* by M. Franck in the *Séances et Travaux de l'Académie des Sciences morales et politiques*, Vol. X. pages 372, 382, and Vol. XI. page 139. It is scarcely necessary to remark that M. Gouraud has gained distinction in other branches of literature since the publication of his work which we have here noticed.

There is one history of our subject which is indeed only a sketch but traced in lines of light by the hand of the great master himself: Laplace devoted a few pages of the introduction to his celebrated work to recording the names of his predecessors and their contributions to the Theory of Probability. It is much to be regretted that he did not supply specific references throughout his treatise, in order to distinguish carefully between that which he merely transmitted from preceding mathematicians and that which he originated himself.

It is necessary to observe that in cases where I point out a similarity between the investigations of two or more writers I do not mean to imply that these investigations could not have been made independently. Such coincidences may occur easily and naturally without any reason for imputing unworthy conduct to those who succeed the author who had the priority in publication. I draw attention to this circumstance because I find with regret that from a passage in my former historical work an inference has been drawn of the kind which I here disclaim. In the case of a writer like Laplace who agrees with his predecessors, not in one or two points but in very many, it is of course obvious that he must have borrowed largely, and we conclude that he supposed the erudition of his contemporaries would be sufficient to prevent them from ascribing to himself more than was justly due.

It will be seen that I have ventured to survey a very extensive field of mathematical research. It has been my aim to estimate carefully and impartially the character and the merit of the numerous memoirs and works which I have examined; my criticism has been intentionally close and searching, but I trust never irreverent nor unjust. I have sometimes explained fully the errors which I detected; sometimes, when the detailed exposition of the error would have required more space than the matter deserved, I have given only a brief indication which may be serviceable to a student of the original production itself. I have not hesitated to introduce remarks and developments of my own whenever the subject seemed to require them. In an elaborate German review of my former publication on mathematical history it was suggested that my own contributions were too prominent, and that the purely historical character of the work was thereby impaired; but I have not been induced to change my plan, for I continue to think that such additions as I have been able to make tend to render the subject more intelligible and more complete, without disturbing in any serious degree the continuity of the history. I cannot venture to expect that in such a difficult subject I shall be quite free from error either in my exposition of the labours of others, or in my own contributions; but I hope that such failures will not be numerous nor important. I shall receive most gratefully intimations of any errors or omissions which may be detected in the work.

I have been careful to corroborate my statements by exact quotations from the originals, and these I have given in the languages in which they were published, instead of translating them; the course which I have here adopted is I understand more agreeable to foreign students into whose hands the book may fall. I have been careful to preserve the historical notices and references which occurred in the works I studied; and by the aid of the *Table of Contents*, the *Chronological List*, and the *Index*, which accompany the present volume, it will be easy to ascertain with regard to any proposed mathematician down to the close of the eighteenth century, whether he has written anything upon the Theory of Probability.

I have carried the history down to the close of the eighteenth century; in the case of Laplace, however, I have passed beyond this limit: but by far the larger part of his labours on the Theory of Probability were accomplished during the eighteenth century, though collected and republished by him in his celebrated work in the early part of the present century, and it was therefore convenient to include a full account of all his researches in the present volume. There is ample scope for a continuation of the work which should conduct the history through the period which has elapsed since the close of the eighteenth century; and I have already made some progress in the analysis of the rich materials. But when I consider the time and labour expended on the present volume, although reluctant to abandon a long cherished design, I feel far less sanguine than once I did that I shall have the leisure to arrive at the termination I originally ventured to propose to myself.

Although I wish the present work to be regarded principally as a history, yet there are two other aspects under which it may solicit the attention of students. It may claim the title of a comprehensive treatise on the Theory of Probability, for it assumes in the reader only so much knowledge as can be gained from an elementary book on Algebra, and introduces him to almost every process and every species of problem which the literature of the subject can furnish; or the work may be considered more specially as a commentary on the celebrated treatise of Laplace,—and perhaps no mathematical treatise ever more required or more deserved such an accompaniment.

My sincere thanks are due to Professor De Morgan, himself conspicuous among cultivators of the Theory of Probability, for the kind interest which he has taken in my work, for the loan of scarce books, and for the suggestion of valuable references. A similar interest was manifested by one prematurely lost to science, whose mathematical and metaphysical genius, attested by his marvellous work on the *Laws of Thought*, led him naturally and rightfully in that direction which Pascal and Leibnitz had marked with the unfading lustre of their approbation; and who by his rare ability, his wide attainments, and his attractive character, gained the affection and the reverence of all who knew him.

I. TODHUNTER.

CAMBRIDGE,
May, 1865.