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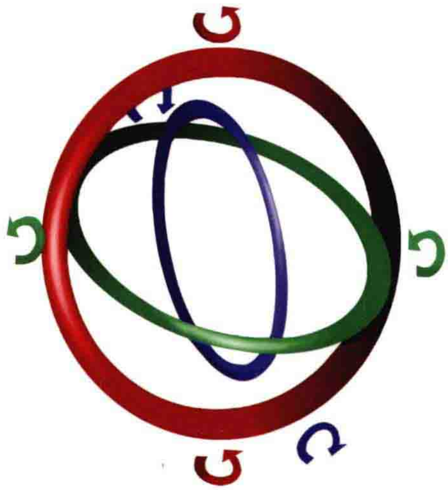
Topics in Quaternion Linear Algebra



Leiba Rodman

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To Ella

Preface

This is probably the first book devoted entirely to linear algebra and matrix analysis over the skew field of real quaternions.

The book is intended for the primary audience of mathematicians working in the area of linear algebra and matrix analysis, instructors and students of these subjects, mathematicians working in related areas such as operator theory and differential equations, researchers who work in other areas and for whom the book is intended as a reference, and scientists (primarily physicists, chemists, and computer scientists) and engineers who may use the book as a reference as well.

The exposition is accessible to upper undergraduate and graduate students in mathematics, science, and engineering. A background in college linear algebra and a modicum of complex analysis and multivariable calculus will suffice.

I intend to keep up with the use of the book. So, I have a request of the readers: please send remarks, corrections, criticism, etc., concerning the book to me at lxrodm@gmail.com or lxrodm@math.wm.edu.

I thank J. Baez for consultation concerning automorphisms of the division algebra of real quaternions, R. M. Guralnick and R. Pereira for consultations concerning invariants for equivalence of matrices over noncommutative principal ideal domains (in particular polynomials with quaternion coefficients), C.-K. Li and Y.-T. Poon for consultations concerning numerical ranges, F. Zhang for helping out with determinants, and V. Bolotnikov. In the final stages of preparation of the manuscript I took advice from several people whose input is greatly appreciated: P. Lancaster, N. J. Higham, H. Schneider, R. Brualdi, and H. J. Woerdeman. I also thank M. Karow and two anonymous reviewers for careful reading of the manuscript and many helpful suggestions.

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Key words: quaternion, quaternion matrix, canonical form, numerical range, similarity, congruence, Smith form, Jordan form, Kronecker form, invariant subspaces, semidefinite subspaces, matrix pencils, hermitian matrix, Hamiltonian matrix, skewhermitian matrix, skew-Hamiltonian matrix, symmetric matrix.

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Contents

Chapter One

Introduction

Besides the introduction, front matter, back matter, and Appendix (Chapter 15), the book consists of two parts. The first part comprises Chapters 2–7. Here, fundamental properties and constructions of linear algebra are explored in the context of quaternions, such as matrix decompositions, numerical ranges, Jordan and Kronecker canonical forms, canonical forms under congruence, determinants, invariant subspaces, etc. The exposition in the first part is on the level of an upper undergraduate or graduate textbook. The second part comprises Chapters 8–14. Here, the emphasis is on canonical forms of quaternion matrix pencils with symmetries or, what is the same, pairs of matrices with symmetries, and the exposition approaches that of a research monograph. Applications are given to systems of linear differential equations with symmetries, and matrix equations.

The mathematical tools used in the book are easily accessible to undergraduates with a background in linear algebra and rudiments of complex analysis and, on occasion, multivariable calculus. The exposition is largely based on tools of matrix analysis. The author strived to make the book self-contained and inclusive of complete proofs as much as possible, at the same time keeping the size of the book within reasonable limits. However, some compromises were inevitable here. Thus, proofs are often omitted for many linear algebra results that are standard for real and complex matrices, are often presented in textbooks, and are valid for quaternion matrices as well with essentially the same proofs.

The book can be used in a variety of ways. More than 200 exercises are provided, on various levels of difficulty, ranging from routine verification of facts and numerical examples designed to illustrate the results to open-ended questions. The exercises and detailed exposition make the book suitable in teaching as supplementary material for undergraduate courses in linear algebra, as well as for students' independent study or reading courses. For students' benefit, several appendices are included that contain background material used in the main text. The book can serve as a basis for a graduate course named advanced linear algebra, topics in linear algebra, or (for those who want to keep the narrower focus) quaternion linear algebra. For example, one can build a graduate course based on Chapters 2–8 and selections from later chapters.

Open problems presented in the book provide an opportunity to do original research. The open problems are on various levels: open-ended problems that may serve as subject for research by mathematicians and concrete, more-specific problems that are perhaps more suited for undergraduate research work under faculty supervision, honors theses, and the like.

For working mathematicians in both theoretical and applied areas, the book may serve as a reference source. Such areas include, in particular, vector calculus, ordinary and partial differential equations, and boundary value problems (see, e.g., Gürlebeck and Sprössig [60]), and numerical analysis (Bunse-Gerstner et al. [22]). The accessibility and importance of the mathematics should make this book a widely useful work not only for mathematicians, but also for scientists and engineers.

Quaternions have become increasingly useful for practitioners in research, both in theory and applications. For example, a significant number of research papers on quaternions, perhaps even most of them, appear regularly in mathematical physics journals, and quantum mechanics based on quaternion analysis is mainstream physics. In engineering, quaternions are often used in control systems, and in computer science they play a role in computer graphics. Quaternion formalism is also used in studies of molecular symmetry. For practitioners in these areas, the book can serve as a valuable reference tool.

New, previously unpublished results presented in the book with complete proofs will hopefully be useful for experts in linear algebra and matrix analysis. Much of the material appears in a book for the first time; this is true for Chapters 5–14, most of Chapter 4, and a substantial part of Chapter 3.

As far as the author is aware, this is the first book dedicated to systematic exposition of quaternion linear algebra. So far, there are only a few expository papers and chapters in books on the subject (for example, Chapter 1 in Gürlebeck and Sprössig [60], Brieskorn [20], Zhang [164], or Farenick and Pidkowich [38]) as well as algebraic treatises on skew fields (e.g., Cohn [29] or Wan [156]).

It is inevitable that many parts of quaternion linear algebra are not reflected in the book, most notably those parts pertaining to numerical analysis (Bunse-Gerstner et al. [22] and Faßbender et al. [40]). Also, the important classes of orthogonal, unitary, and symplectic quaternion matrices are given only brief exposure.

We now describe briefly the contents of the book chapter by chapter.

Chapter 2 concerns (scalar) quaternions and the basic properties of quaternion algebra, with emphasis on solution of equations such as $axb = c$ and $ax - xb = c$. Description of all automorphisms and antiautomoprhisms of quaternions is given, and representations of quaternions in terms of 2×2 complex matrices and 4×4 real matrices are introduced. These representations will play an important role throughout the book.

Chapter 3 covers basics on the vector space of columns with quaternion components, matrix algebra, and various matrix decomposition. The real and complex representations of quaternions are extended to vectors and matrices. Various matrix decompositions are studied; in particular, Cholesky factorization is proved for matrices that are hermitian with respect to involutions other than the conjugation. A large part of this chapter is devoted to numerical ranges of quaternion matrices with respect to conjugation as well as with respect to other involutions. Finally, a brief exposition is given for the set of quaternion subspaces, understood as a metric space with respect to the metric induced by the gap function.

In a short Chapter 4 we develop diagonal canonical forms and prove inertia theorems for hermitian and skewhermitian matrices with respect to involutions (including the conjugation). We also identify dimensions of subspaces that are neutral or semidefinite relative to a given hermitian matrix and are maximal with respect to this property. The material in Chapters 3 and 4 does not depend on the more involved constructions such as the Jordan form and its proof.

Chapter 5 is a key chapter in the book. Root subspaces of quaternion matrices are introduced and studied. The Jordan form of a quaternion matrix is presented in full detail, including a complete proof. The complex matrix representation plays a crucial role here. Although the standard definition of a determinant