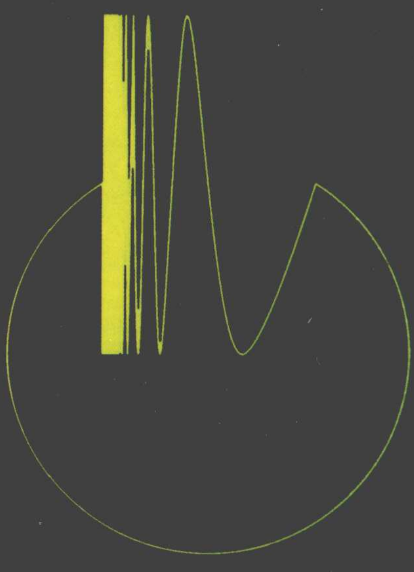
天元基金影印数学丛书

Real Mathematical Analysis

实数学分析（影印版）

Charles Chapman Pugh



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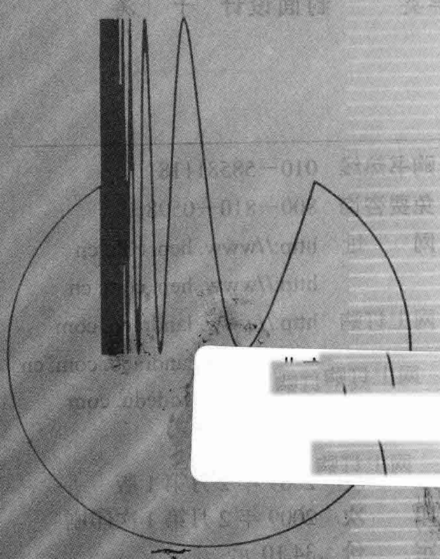

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序言

为了更好地借鉴国外数学教育与研究的成功经验，促进我国数学教育与研究事业的发展，提高高等学校数学教育教学质量，本着“为我国热爱数学的青年创造一个较好的学习数学的环境”这一宗旨，天元基金赞助出版“天元基金影印数学丛书”。

该丛书主要包含国外反映近代数学发展的纯数学与应用数学方面的优秀书籍，天元基金邀请国内各个方向的知名数学家参与选题的工作，经专家遴选、推荐，由高等教育出版社影印出版。为了提高我国数学研究生教学的水平，暂把选书的目标确定在研究生教材上。当然，有的书也可作为高年级本科生教材或参考书，有的书则介于研究生教材与专著之间。

欢迎各方专家、读者对本丛书的选题、印刷、销售等工作提出批评和建议。

天元基金领导小组

2007 年 1 月

To the students who have encouraged me
—especially A.W., D.H., and M.B.

Preface

Was plane geometry your favorite math course in high school? Did you like proving theorems? Are you sick of memorizing integrals? If so, real analysis could be your cup of tea. In contrast to calculus and elementary algebra, it involves neither formula manipulation nor applications to other fields of science. None. It is pure mathematics, and I hope it appeals to you, the budding pure mathematician.

Berkeley, California, USA CHARLES CHAPMAN PUGH

Contents

1

Real Numbers

1 Preliminaries

Before discussing the system of real numbers it is best to make a few general remarks about mathematical outlook.

Language

By and large, mathematics is expressed in the language of set theory. Your first order of business is to get familiar with its vocabulary and grammar. A set is a collection of elements. The elements are members of the set and are said to belong to the set. For example, $\mathbb{N}$ denotes the set of **natural numbers**, $1, 2, 3, \ldots$. The members of $\mathbb{N}$ are whole numbers greater than or equal to 1. Is 10 a member of $\mathbb{N}$? Yes, 10 belongs to $\mathbb{N}$. Is 0 a member of $\mathbb{N}$? No. We write

$$x \in A \quad \text{and} \quad y \notin B$$

to indicate that the element x is a member of the set A and y is not a member of B. Thus, $6819 \in \mathbb{N}$ and $0 \notin \mathbb{N}$.

We try to write capital letters for sets and small letters for elements of sets. Other standard sets have standard names. The set of **integers** is denoted by $\mathbb{Z}$, which stands for the German word *zahlen*. (An integer is a positive whole number, zero, or a negative whole number.) Is $\sqrt{2} \in \mathbb{Z}$? No, $\sqrt{2} \notin \mathbb{Z}$. How about -15? Yes, $-15 \in \mathbb{Z}$.

The set of **rational numbers** is called $\mathbb{Q}$, which stands for "quotient." (A rational number is a fraction of integers, the denominator being nonzero.) Is $\sqrt{2}$ a member of $\mathbb{Q}$? No, $\sqrt{2}$ does not belong to $\mathbb{Q}$. Is π a member of $\mathbb{Q}$? No. Is 1.414 a member of $\mathbb{Q}$? Yes.

You should practice reading the notation "$\{x \in A :$" as "the set of x that belong to A such that." The **empty set** is the collection of no elements and is denoted by $\emptyset$. Is 0 a member of the empty set? No, $0 \notin \emptyset$.

A **singleton set** has exactly one member. It is denoted as $\{x\}$ where x is the member. Similarly if exactly two elements x and y belong to a set, the set is denoted as $\{x, y\}$.

If A and B are sets and each member of A also belongs to B then A is a subset of B and A is contained in B. We write[†]

$$A \subset B.$$

Is $\mathbb{N}$ a subset of $\mathbb{Z}$? Yes. Is it a subset of $\mathbb{Q}$? Yes. If A is a subset of B and B is a subset of C, does it follow that A is a subset of C? Yes. Is the empty set a subset of $\mathbb{N}$? Yes, $\emptyset \subset \mathbb{N}$. Is 1 a subset of $\mathbb{N}$? No, but the singleton set $\{1\}$ is a subset of $\mathbb{N}$. Two sets are equal if each member of one belongs to the other. Each is a subset of the other. This is how you prove two sets are equal: show that each element of the first belongs to the second, and each element of the second belongs to the first.

The union of the sets A and B is the set $A \cup B$, each of whose elements belongs to either A, or to B, or to both A and to B. The intersection of A and B is the set $A \cap B$ each of whose elements belongs to both A and to B. If $A \cap B$ is the empty set then A and B are **disjoint**. The **symmetric difference** of A and B is the set $A \Delta B$ each of whose elements belongs to A but not to B, or belongs to B but not to A. The **difference** of A to B is the set $A \setminus B$ whose elements belong to A but not to B. See Figure 1.

A **class** is a collection of sets. The sets are members of the class. For example we could consider the class $\mathcal{E}$ of sets of even natural numbers. Is the set $\{2, 15\}$ a member of $\mathcal{E}$? No. How about the singleton set $\{6\}$? Yes. How about the empty set? Yes, each element of the empty set is even.

When is one class a subclass of another? When each member of the former belongs also to the latter. For example the class $\mathcal{T}$ of sets of positive integers divisible by 10 is a subclass of $\mathcal{E}$, the class of sets of even natural

[†] When some mathematicians write $A \subset B$ they mean that A is a subset of B, but $A \neq B$. We do *not* adopt this convention. We accept $A \subset A$.

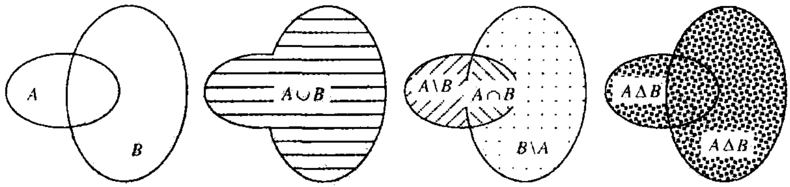

Figure 1 Venn diagrams of union, intersection, and differences.

numbers, and we write $\mathcal{T} \subset \mathcal{E}$. Each set that belongs to the class $\mathcal{T}$ also belongs to the class $\mathcal{E}$. Consider another example. Let $\mathcal{S}$ be the class of singleton subsets of $\mathbb{N}$ and $\mathcal{D}$ be the class of subsets of $\mathbb{N}$ each of which has exactly two elements. Thus $\{10\} \in \mathcal{S}$ and $\{2, 6\} \in \mathcal{D}$. Is $\mathcal{S}$ a subclass of $\mathcal{D}$? No. The members of $\mathcal{S}$ are singleton sets and they are not members of $\mathcal{D}$. Rather they are subsets of members of $\mathcal{D}$. Note the distinction, and think about it.

Here is an analogy. Each citizen is a member of his or her country – I am an element of the USA and Tony Blair is an element of the UK. Each country is a member of the United Nations. Are citizens members of the UN? No, countries are members of the UN.

In the same vein is the concept of an **equivalence relation** on a set S. It is a relation $s \sim s'$ that holds between some members $s, s' \in S$ and it satisfies three properties: For all $s, s', s'' \in S$

(a) $s \sim s$.

(b) $s \sim s'$ implies that $s' \sim s$.

(c) $s \sim s' \sim s''$ implies that $s \sim s''$.

The equivalence relation breaks S into disjoint subsets called **equivalence classes**† defined by mutual equivalence: the equivalence class containing s consists of all elements $s' \in S$ equivalent to s and is denoted $[s]$. The element s is a **representative** of its equivalence class. See Figure 2. Think again of citizens and countries. Say two citizens are equivalent if they are citizens of the same country. The world of equivalence relations is egalitarian: I represent my equivalence class USA just as much as does the President.

Truth

When is a mathematical statement accepted as true? Generally, mathematicians would answer "Only when it has a proof inside a familiar mathematical

† The phrase "equivalence class" is standard and widespread, although it would be more consistent with the idea that a class is a collection of sets to refer instead to an "equivalence set."

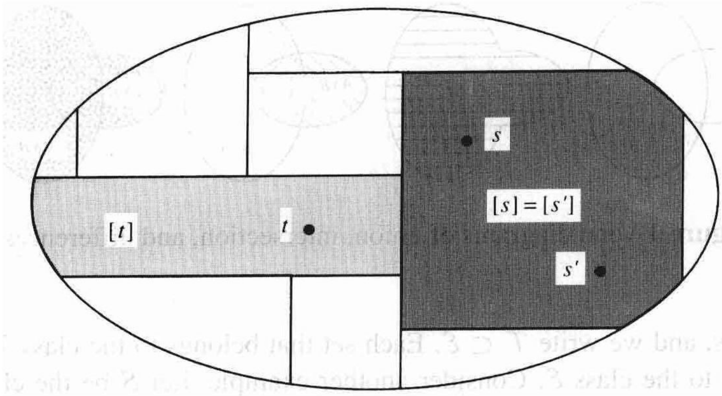

Figure 2 Equivalence classes and representatives.

framework." A picture may be vital in getting you to believe a statement. An analogy with something you know to be true may help you understand it. An authoritative teacher may force you to parrot it. A formal proof, however, is the ultimate and only reason to accept a mathematical statement as true. A recent debate in Berkeley focused the issue for me. According to a math teacher from one of our local private high schools, his students found proofs in mathematics were of little value, especially compared to "convincing arguments." Besides, the mathematical statements were often seen as obviously true and in no need of formal proof anyway. I offer you a paraphrase of Bob Osserman's response.

> But a convincing argument is not a proof. A mathematician generally wants both, and certainly would be less likely to accept a convincing argument by itself than a formal proof by itself. Least of all would a mathematician accept the proposal that we should generally replace proofs with convincing arguments.
>
> There has been a tendency in recent years to take the notion of proof down from its pedestal. Critics point out that standards of rigor change from century to century. New gray areas appear all the time. Is a proof by computer an acceptable proof? Is a proof that is spread over many journals and thousands of pages, that is too long for any one person to master, a proof? And of course, venerable Euclid is full of flaws, some filled in by Hilbert, others possibly still lurking.

> Clearly it is worth examining closely and critically the most basic notion of mathematics, that of proof. On the other hand, it is important to bear in mind that all distinctions and niceties about what precisely constitutes a proof are mere quibbles compared to the enormous gap between any generally accepted version of a proof and the notion of a convincing argument. Compare Euclid, with all his flaws to the most eminent of the ancient exponents of the convincing argument — Aristotle. Much of Aristotle's reasoning was brilliant, and he certainly convinced most thoughtful people for over a thousand years. In some cases his analyses were exactly right, but in others, such as heavy objects falling faster than light ones, they turned out to be totally wrong. In contrast, there is not to my knowledge a single theorem stated in Euclid's *Elements* that in the course of two thousand years turned out to be false. That is quite an astonishing record, and an extraordinary validation of proof over convincing argument.

Here are some guidelines for writing a rigorous mathematical proof. See also Exercise 0.

1. Name each object that appears in your proof. (For instance, you might begin your proof with a phrase, "consider a set X, and elements x, y that belong to X," etc.)
2. Draw a diagram that captures how these objects relate, and extract logical statements from it. Quantifiers precede the objects quantified; see below.
3. Proceed step-by-step, each step depending on the hypotheses, previously proved theorems, or previous steps in your proof.
4. Check for "rigor": all cases have been considered, all details have been tied down, and circular reasoning has been avoided.
5. Before you sign off on the proof, check for counter-examples and any implicit assumptions you made that could invalidate your reasoning.

Logic

Among the most frequently used logical symbols in math are the quantifiers $\forall$ and $\exists$. Read them always as "for each" and "there exists." Avoid reading $\forall$ as "for all," which in English has a more inclusive connotation. Another common symbol is $\implies$. Read it as "implies."

The rules of correct mathematical grammar are simple: quantifiers appear at the beginning of a sentence, they modify only what follows them in the sentence, and assertions occur at the end of the sentence. Here is an example.

(1)
For each integer n there is a prime number p which is greater than n.

In symbols the sentence reads

$$\forall n \in \mathbb{Z} \quad \exists p \in P \quad \text{such that} \quad p > n,$$

where P denotes the set of prime numbers. (A **prime number** is a whole number greater than 1 whose only divisors in $\mathbb{N}$ are itself and 1.) In English, the same idea can be re-expressed as

(2) *Every integer is less than some prime number*

or

(3) *A prime number can always be found which is greater than any given integer.*

These sentences are correct in English grammar, but disastrously WRONG when transcribed directly into mathematical grammar. They translate into disgusting mathematical gibberish:

(WRONG 2) $\forall n \in \mathbb{Z} \quad n < p \quad \exists p \in P$

(WRONG 3) $\exists p \in P \quad p > n \quad \forall n \in \mathbb{Z}.$

Moral Quantifiers first and assertions last. In stating a theorem, try to apply the same principle. Write the hypothesis first and the conclusion second. See Exercise 0.

The order in which quantifiers appear is also important. Contrast the next two sentences in which we switch the position of two quantified phrases.

(4) $(\forall n \in \mathbb{N}) \quad (\forall m \in \mathbb{N}) \quad (\exists p \in P) \quad \text{such that} \quad (nm < p).$

(5) $(\forall n \in \mathbb{N}) \quad (\exists p \in P) \quad \text{such that} \quad (\forall m \in \mathbb{N}) \quad (nm < p).$

(4) is a true statement but (5) is false. A quantifier modifies the part of a sentence that follows it but not the part that precedes it. This is another reason never to end with a quantifier.

Moral Quantifier order is crucial.

There is a point at which English and mathematical meaning diverge. It concerns the word "or." In mathematics "a or b" always means "a or b or both a and b," while in English it can mean "a or b but not both a and b." For example, Patrick Henry certainly would not have accepted both liberty and death in response to his cry of "Give me liberty or give me death." In mathematics, however, the sentence "17 is a prime or 23 is a prime" is correct even though both 17 and 23 are prime. Similarly, in mathematics a $\Rightarrow b$ means that if a is true then b is true but that b might also be true for reasons entirely unrelated to the truth of a. In English, $a \Rightarrow b$ is often confused with $b \Rightarrow a$.

Moral In mathematics, "or" is inclusive. It means *and/or*. In mathematics, $a \Rightarrow b$ is not the same as $b \Rightarrow a$.

It is often useful to form the negation or logical opposite of a mathematical sentence. The symbol $\sim$ is usually used for negation, despite the fact that the same symbol also indicates an equivalence relation. Mathematicians refer to this as an **abuse of notation**. Fighting a losing battle against abuse of notation, we write $\neg$ for negation. For example, if $m, n \in \mathbb{N}$ then $\neg(m < n)$ means it is not true that m is less than n. In other words

$$\neg(m < n) \quad \equiv \quad m \geq n.$$

(We use the symbol $\equiv$ to indicate that the two statements are equivalent.) Similarly, $\neg(x \in A)$ means it is not true that x belongs to A. In other words,

$$\neg(x \in A) \quad \equiv \quad x \notin A.$$

Double negation returns a statement to its original meaning. Slightly more interesting is the negation of "and" and "or." Just for now, let us use the symbols & for "and" and $\vee$ for "or." We claim

(6) $$\neg(a \,\&\, b) \quad \equiv \quad \neg a \vee \neg b.$$

(7) $$\neg(a \vee b) \quad \equiv \quad \neg a \,\&\, \neg b.$$

For if it is not the case that both a and b are true then at least one must be false. This proves (6), and (7) is similar. Implication also has such interpretations:

(8) $$a \Rightarrow b \quad \equiv \quad \neg a \Leftarrow \neg b \quad \equiv \quad \neg a \vee b.$$

$$(9) \qquad \neg(a \Rightarrow b) \quad \equiv \quad a \;\&\; \neg b.$$

What about the negation of a quantified sentence such as

$$\neg(\forall n \in \mathbb{N}, \exists p \in P \text{ such that } n < p).$$

The rule is: change each $\forall$ to $\exists$ and vice versa, leaving the order the same, and negate the assertion. In this case the negation is

$$\exists n \in \mathbb{N}, \quad \forall p \in P, \quad n \geq p.$$

In English it reads "There exists a natural number n, and for all primes p, $n \geq p$." The sentence has correct mathematical grammar but of course is false. To help translate from mathematics to readable English, a comma can be read as "and" or "such that."

All mathematical assertions take an implication form $a \Rightarrow b$. The hypothesis is a and the conclusion is b. If you are asked to prove $a \Rightarrow b$, there are several ways to proceed. First you may just see right away why a does imply b. Fine, if you are so lucky. Or you may be puzzled. Does a really imply b? Two routes are open to you. You may view the implication in its equivalent contrapositive form $\neg a \Leftarrow \neg b$ as in (8). Sometimes this will make things clearer. Or you may explore the possibility that a fails to imply b. If you can somehow deduce from the failure of a implying b a contradiction to a known fact (for instance if you can deduce the existence of a planar right triangle with legs x, y but $x^2 + y^2 \neq h^2$ where h is the hypotenuse) then you have succeeded in making an **argument by contradiction**. Clearly (9) is pertinent here. It tells you what it means that a fails to imply b, namely that a is true and, simultaneously, b is false.

Euclid's proof that $\mathbb{N}$ contains infinitely many prime numbers, is a classic example of this method. The hypothesis is that $\mathbb{N}$ is the set of natural numbers and that P is the set of prime numbers. The conclusion is that P is an infinite set. The proof of this fact begins with the phrase "Suppose not." It means: suppose, after all, that the set of prime numbers P is merely a finite set, and see where this leads you. It does not mean that we think P really is a finite set, and it is not a hypothesis of a theorem. Rather it just means that we will try to find out what awful consequences would follow from P being finite. In fact if P were† finite then it would consist of m

†In English grammar, the subjunctive mode indicates doubt, and I have written Euclid's proof in that form – "if P *were* finite" instead of "if P *is* finite," "each prime *would* divide N evenly," instead of "each prime *divides* N evenly," etc. At first it seems like a fine idea to write all arguments by contradiction in the subjunctive mode, exhibiting clearly their impermanence. Soon, however, the subjunctive and conditional language becomes ridiculously stilted and archaic. For consistency then, as much as possible, *use the present tense.*