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COMBINATORIAL MATRIX CLASSES

Richard A. Brualdi

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Combinatorial Matrix Classes

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Preface

In the preface of the book *Combinatorial Matrix Theory*[1] (CMT) I discussed my plan to write a second volume entitled *Combinatorial Matrix Classes*. Here 15 years later (including 6, to my mind, wonderful years as Department of Mathematics Chair at UW-Madison), and to my great relief, is the finished product. What I proposed as topics to be covered in a second volume were, in retrospect, much too ambitious. Indeed, after some distance from the first volume, it now seems like a plan for a book series rather than for a second volume. I decided to concentrate on topics that I was most familiar with and that have been a source of much research inspiration for me. Having made this decision, there was more than enough basic material to be covered. Most of the material in the book has never appeared in book form, and as a result, I hope that it will be useful to both current researchers and aspirant researchers in the field. I have tried to be as complete as possible with those matrix classes that I have treated, and thus I also hope that the book will be a useful reference book.

I started the serious writing of this book in the summer of 2000 and continued, while on sabbatical, through the following semester. I made good progress during those six months. Thereafter, with my many teaching, research, editorial, and other professional and university responsibilities, I managed to work on the book only sporadically. But after 5 years, I was able to complete it or, if one considers the topics mentioned in the preface of CMT, one might say I simply stopped writing. But that is not the way I feel. I think, and I hope others will agree, that the collection of matrix classes developed in the book fit together nicely and indeed form a coherent whole with no glaring omissions. Except for a few reference to CMT, the book is self-contained.

My primary inspiration for combinatorial matrix classes has come from two important contributors, Herb Ryser and Ray Fulkerson. In a real sense, with their seminal and early research, they are the "fathers" of the subject. Herb Ryser was my thesis advisor and I first learned about the class $\mathcal{A}(R, S)$, which occupies a very prominent place in this book, in the fall of 1962 when I was a graduate student at Syracuse University (New York).

[1] Authored by Richard A. Brualdi and Herbert J. Ryser and published by Cambridge University Press in 1991.

In addition, some very famous mathematicians have made seminal contributions that have directly or indirectly impacted the study of matrix classes. With the great risk of offending someone, let me mention only Claude Berge, Garrett Birkhoff, David Gale, Alan Hoffman, D. König, Victor Klee, Donald Knuth, H.G. Landau, Leon Mirsky, and Bill Tutte. To these people, and all others who have contributed, I bow my head and say a heartfelt thank-you for your inspiration.

As I write this preface in the summer of 2005, I have just finished my 40th year as a member of the Department of Mathematics of the University of Wisconsin in Madison. I have been fortunate in my career to be a member of a very congenial department that, by virtue of its faculty and staff, provides such a wonderful atmosphere in which to work, and that takes teaching, research, and service all very seriously. It has also been my good fortune to have collaborated with my graduate students, and postdoctoral fellows, over the years, many of whom have contributed to one or more of the matrix classes treated in this book. I am indebted to Geir Dahl who read a good portion of this book and provided me with valuable comments.

My biggest source of support these last 10 years has been my wife Mona. Her encouragement and love have been so important to me.

Richard A. Brualdi
Madison, Wisconsin

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1

Introduction

In this chapter we introduce some concepts and theorems that are important for the rest of this book. Much, but not all, of this material can be found in the book [4]. In general, we have included proofs of theorems only when they do not appear in [4]. The proof of Theorem 1.7.1 is an exception, since we give here a much different proof. We have not included all the basic terminology that we make use of (e.g. graph-theoretic terminology), expecting the reader either to be familiar with such terminology or to consult [4] or other standard references.

1.1 Fundamental Concepts

Let

$$A = [a_{ij}] \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

be a matrix of m rows and n columns. We say that A is of *size* m by n, and we also refer to A as an m by n matrix. If $m = n$, then A is a *square* matrix of *order* n. The elements of the matrix A are always real numbers and usually are nonnegative real numbers. In fact, the elements are sometimes restricted to be nonnegative integers, and often they are restricted to be either 0 or 1. The matrix A is composed of m row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ and n column vectors $\beta_1, \beta_2, \ldots, \beta_n$, and we write

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = [\beta_1\, \beta_2\, \ldots\, \beta_n] .$$

It is sometimes convenient to refer to either a row or column of the matrix A as a *line* of A. We use the notation A^{T} for the transpose of the matrix A. If $A = A^{\mathrm{T}}$, then A is a square matrix and is *symmetric*.

A zero matrix is always designated by O, a matrix with every entry equal to 1 by J, and an identity matrix by I. In order to emphasize the size of these matrices we sometimes include subscripts. Thus $J_{m,n}$ denotes the m by n matrix of all 1's, and this is shortened to J_n if $m = n$. The notations $O_{m,n}, O_n,$ and I_n have similar meanings.

A *submatrix* of A is specified by choosing a subset of the row index set of A and a subset of the column index set of A. Let $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$. Let $\bar{I} = \{1, 2, \ldots, m\} \setminus I$ denote the *complement* of I in $\{1, 2, \ldots, m\}$, and let $\bar{J} = \{1, 2, \ldots, n\} \setminus J$ denote the complement of J in $\{1, 2, \ldots, n\}$. Then we use the following notations to denote submatrices of A:

$$\begin{aligned}
A[I, J] &= [a_{ij}\colon i \in I, j \in J], \\
A(I, J] &= [a_{ij}\colon i \in \bar{I}, j \in J], \\
A[I, J) &= [a_{ij}\colon i \in I, j \in \bar{J}], \\
A(I, J) &= [a_{ij}\colon i \in \bar{I}, j \in \bar{J}], \\
A[I, \cdot] &= A[I, \{1, 2, \ldots, n\}], \\
A[\cdot, J] &= A[\{1, 2, \ldots, m\}, J], \\
A(I, \cdot] &= A[\bar{I}, \{1, 2, \ldots, n\}], \text{ and} \\
A[\cdot, J) &= A[\{1, 2, \ldots, m\}, \bar{J}].
\end{aligned}$$

These submatrices are allowed to be empty. If $I = \{i\}$ and $J = \{j\}$, then we abbreviate $A(I, J)$ by $A(i, j)$.

We have the following partitioned forms of A:

$$A = \left[\begin{array}{c|c} A[I, J] & A[I, J) \\ \hline A(I, J] & A(I, J) \end{array}\right], \quad A = \left[\begin{array}{c} A[I, \cdot] \\ \hline A(I, \cdot] \end{array}\right],$$

and

$$A = \left[\begin{array}{c|c} A[\cdot, J] & A[\cdot, J) \end{array}\right].$$

The $n!$ permutation matrices of order n are obtained from I_n by arbitrary permutations of its rows (or of its columns). Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be a permutation of $\{1, 2, \ldots, n\}$. Then π corresponds to the permutation matrix $P_\pi = [p_{ij}]$ of order n in which $p_{i\pi_i} = 1$ $(i = 1, 2, \ldots, n)$ and all other $p_{ij} = 0$. The permutation matrix corresponding to the inverse π^{-1} of

π is P_π^{T}. It thus follows that $P_\pi^{-1} = P_\pi^{\mathrm{T}}$, and thus an arbitrary permutation matrix P of order n satisfies the matrix equation

$$PP^{\mathrm{T}} = P^{\mathrm{T}}P = I_n.$$

Let A be a square matrix of order n. Then the matrix PAP^{T} is similar to A. If we let Q be the permutation matrix P^{T}, then $PAP^{\mathrm{T}} = Q^{\mathrm{T}}AQ$. The row vectors of the matrix $P_\pi A$ are $\alpha_{\pi_1}, \alpha_{\pi_2}, \ldots, \alpha_{\pi_m}$. The column vectors of AP_π are $\beta_{\pi'_1}, \beta_{\pi'_2}, \ldots, \beta_{\pi'_n}$ where $\pi^{-1} = (\pi'_1, \pi'_2, \ldots, \pi'_n)$. The column vectors of AP^{T} are $\beta_{\pi_1}, \beta_{\pi_2}, \ldots, \beta_{\pi_n}$. Thus if P is a permutation matrix, the matrix PAP^{T} is obtained from A by *simultaneous permutations of its rows and columns.* More generally, if A is an m by n matrix and P and Q are permutation matrices of orders m and n, respectively, then the matrix PAQ is a matrix obtained from A by *arbitrary permutations of its rows and columns.*

Let $A = [a_{ij}]$ be a matrix of size m by n. The *pattern* (or *nonzero pattern*) of A is the set

$$\mathcal{P}(A) = \{(i,j) \colon a_{ij} \neq 0, \quad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n\}$$

of positions of A containing a nonzero element.

With the m by n matrix $A = [a_{ij}]$ we associate a combinatorial configuration that depends only on the pattern of A. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a nonempty set of n elements. We call X an *n-set.* Let

$$X_i = \{x_j \colon a_{ij} \neq 0, j = 1,2,\ldots,n\} \quad (i = 1,2,\ldots,m).$$

The collection of m not necessarily distinct subsets $X_1, X_2, \ldots, X_m$ of the n-set X is the *configuration* associated with A. If P and Q are permutation matrices of orders m and n, respectively, then the configuration associated with PAQ is obtained from the configuration associated with A by relabeling the elements of X and reordering the sets $X_1, X_2, \ldots, X_m$. Conversely, given a nonempty configuration $X_1, X_2, \ldots, X_m$ of m subsets of the nonempty n-set $X = \{x_1, x_2, \ldots, x_n\}$, we associate an m by n matrix $A = [a_{ij}]$ of 0's and 1's, where $a_{ij} = 1$ if and only if $x_j \in X_i$ $(i = 1,2,\ldots,m; j = 1,2,\ldots,n)$.

The configuration associated with the m by n matrix $A = [a_{ij}]$ furnishes a particular way to represent the structure of the nonzeros of A. We may view a configuration as a *hypergraph* [1] with vertex set X and *hyperedges* $X_1, X_2, \ldots, X_m$. This hypergraph may have repeated edges, that is, two or more hyperedges may be composed of the same set of vertices. The *edge–vertex incidence matrix* of a hypergraph H with vertex set $X = \{x_1, x_2, \ldots, x_n\}$ and edges $X_1, X_2, \ldots, X_m$ is the m by n matrix $A = [a_{ij}]$ of 0's and 1's in which $a_{ij} = 1$ if and only if x_j is a vertex of edge X_i $(i = 1,2,\ldots,m; j = 1,2,\ldots,n)$. Notice that the hypergraph (configuration) associated with A is the original hypergraph H. If A has exactly two 1's in each row, then A is the edge–vertex incidence matrix of a *multigraph* where a pair of distinct vertices may be joined by more than one edge. If no two rows of A are identical, then this multigraph is a *graph*.

Another way to represent the structure of the nonzeros of a matrix is by a bipartite graph. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be sets of cardinality m and n, respectively, such that $U \cap W = \emptyset$. The *bipartite graph* associated with A is the graph BG(A) with vertex set $V = U \cup W$ whose edges are all the pairs $\{u_i, w_j\}$ for which $a_{ij} \neq 0$. The pair $\{U, W\}$ is the *bipartition* of BG(A).

Now assume that A is a nonnegative integral matrix, that is, the elements of A are nonnegative integers. We may then associate with A a *bipartite multigraph* BMG(A) with the same vertex set V bipartitioned as above into U and W. In BMG(A) there are a_{ij} edges of the form $\{u_i, w_j\}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Notice that if A is a (0,1)-matrix, that is, each entry is either a 0 or a 1, then the bipartite multigraph BMG(A) is a bipartite graph and coincides with the bipartite graph BG(A). Conversely, let BMG be a bipartite multigraph with bipartitioned vertex set $V = \{U, W\}$ where U and W are as above. The *bipartite adjacency matrix* of BMG, abbreviated *bi-adjacency matrix*, is the m by n matrix $A = [a_{ij}]$ where a_{ij} equals the number of edges of the form $\{u_i, w_j\}$ (the *multiplicity* of (u_i, v_j)) $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Notice that BMG($A$) is the original bipartite multigraph BMG.

An m by n matrix A is called *decomposable* provided there exist nonnegative integers p and q with $0 < p+q < m+n$ and permutation matrices P and Q such that PAQ is a direct sum $A_1 \oplus A_2$ where A_1 is of size p by q. The conditions on p and q imply that the matrices A_1 and A_2 may be vacuous[1] but each of them contains either a row or a column. The matrix A is *indecomposable* provided it is not decomposable. *The bipartite graph* BG(A) *is connected if and only if A is is indecomposable.*

Assume that the matrix $A = [a_{ij}]$ is square of order n. We may represent its nonzero structure by a digraph $D(A)$. The vertex set of $D(A)$ is taken to be an n-set $V = \{v_1, v_2, \ldots, v_n\}$. There is an arc (v_i, v_j) from v_i to v_j if and only if $a_{ij} \neq 0$ $(i, j = 1, 2, \ldots, n)$. Notice that a nonzero diagonal entry of A determines an arc of $D(A)$ from a vertex to itself (a *directed loop* or *di-loop*). If A is, in addition, a nonnegative integral matrix, then we associate with A a *general digraph* GD(A) with vertex set V where there are a_{ij} arcs of the form (v_i, v_j) $(i, j = 1, 2, \ldots, n)$. If A is a (0,1)-matrix, then GD(A) is a digraph and coincides with $D(A)$. Conversely, let GD be a general digraph with vertex set V. The *adjacency matrix* of GD(A) is the nonnegative integral matrix $A = [a_{ij}]$ of order n where a_{ij} equals the number of arcs of the form (v_i, v_j) (the *multiplicity* of (v_i, v_j)) $(i, j = 1, 2, \ldots, n)$. Notice that GD(A) is the original general digraph GD.

Now assume that the matrix $A = [a_{ij}]$ not only is square but is also symmetric. Then we may represent its nonzero structure by the *graph*

[1] If $p + q = 1$, then A_1 has either a row but no columns or a column but no rows. A similar conclusion holds for A_2 if $p + q = m + n - 1$.

$G(A)$. The vertex set of $G(A)$ is an n-set $V = \{v_1, v_2, \ldots, v_n\}$. There is an edge joining v_i and v_j if and only if $a_{ij} \neq 0$ $(i, j = 1, 2, \ldots, n)$. A nonzero diagonal entry of A determines an edge joining a vertex to itself, that is, a *loop*. The graph $G(A)$ can be obtained from the bipartite graph $\mathrm{BG}(A)$ by identifying the vertices u_i and w_i and calling the resulting vertex v_i $(i = 1, 2, \ldots, n)$. If A is, in addition, a nonnegative integral matrix, then we associate with A a *general graph* $\mathrm{GG}(A)$ with vertex set V where there are a_{ij} edges of the form $\{v_i, v_j\}$ $(i, j = 1, 2, \ldots, n)$. If A is a (0,1)-matrix, then $\mathrm{GG}(A)$ is a graph and coincides with $G(A)$. Conversely, let GG be a general graph with vertex set V. The *adjacency matrix* of GG is the nonnegative integral symmetric matrix $A = [a_{ij}]$ of order n where a_{ij} equals the number of edges of the form (v_i, v_j) (the *multiplicity* of $\{v_i, v_j\}$) $(i, j = 1, 2, \ldots, n)$. Notice that $\mathrm{GG}(A)$ is the original general graph GG. A general graph with no loops is called a *multigraph*.

The symmetric matrix A of order n is *symmetrically decomposable* provided there exists a permutation matrix P such that $PAP^{\mathrm{T}} = A_1 \oplus A_2$ where A_1 and A_2 are both matrices of order at least 1; if A is not symmetrically decomposable, then A is *symmetrically indecomposable*. The matrix A is symmetrically indecomposable if and only if its graph $G(A)$ is connected.

Finally, we remark that if a multigraph MG is bipartite with vertex bipartition $\{U, W\}$ and A is the adjacency matrix of MG, then there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} O & C \\ C^{\mathrm{T}} & O \end{bmatrix}$$

where C is the bi-adjacency matrix of MG (with respect to the bipartition $\{U, W\}$).[2]

We shall make use of elementary concepts and results from the theory of graphs and digraphs. We refer to [4], or books on graphs and digraphs, such as [17], [18], [2], [1], for more information.

1.2 Combinatorial Parameters

In this section we introduce several combinatorial parameters associated with matrices and review some of their basic properties. In general, by a *combinatorial property or parameter of a matrix* we mean a property or parameter which is invariant under arbitrary permutations of the rows and columns of the matrix. More information about some of these parameters can be found in [4].

Let $A = [a_{ij}]$ be an m by n matrix. The *term rank* of A is the maximal number $\rho = \rho(A)$ of nonzero elements of A with no two of these elements on a line. The *covering number* of A is the minimal number $\kappa = \kappa(A)$ of

[2] If G is connected, then the bipartition is unique.

lines of A that contain (that is, *cover*) all the nonzero elements of A. Both ρ and κ are combinatorial parameters. The fundamental minimax theorem of König (see [4]) asserts the equality of these two parameters.

Theorem 1.2.1

$$\rho(A) = \kappa(A).$$

A set of nonzero elements of A with no two on a line corresponds in the bipartite graph BG(A) to a set of edges no two of which have a common vertex, that is, pairwise vertex-disjoint edges or a *matching*. Thus Theorem 1.2.1 asserts that in a bipartite graph, the maximal number of edges in a matching equals the minimal number of vertices in a subset of the vertex set that meets all edges.

Assume that $m \leq n$. The *permanent* of A is defined by

$$\mathrm{per}(A) = \sum a_{1i_1} a_{2i_2} \dots a_{mi_m}$$

where the summation extends over all sequences $i_1, i_2, \dots, i_m$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. Thus per($A$) equals the sum of all possible products of m elements of A with the property that the elements in each of the products occur on different lines. The permanent of A is invariant under arbitrary permutations of rows and columns of A, that is,

$$\mathrm{per}(PAQ) = \mathrm{per}(A), \text{if } P \text{ and } Q \text{ are permutation matrices.}$$

If A is a nonnegative matrix, then $\mathrm{per}(A) > 0$ if and only if $\rho(A) = m$. Thus by Theorem 1.2.1, $\mathrm{per}(A) = 0$ if and only if there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & O_{k,l} \\ A_{21} & A_2 \end{bmatrix}$$

for some positive integers k and l with $k + l = n + 1$. In the case of a square matrix, the permanent function is the same as the determinant function apart from a factor ± 1 preceding each of the products in the defining summation. Unlike the determinant, the permanent is, in general, altered by the addition of a multiple of one row to another and the multiplicative law for the determinant, $\det(AB) = \det(A)\det(B)$, does not hold for the permanent. However, the *Laplace expansion* of the permanent by a row or column does hold:

$$\mathrm{per}(A) = \sum_{j=1}^{n} a_{ij} \mathrm{per}(A(i,j)) \quad (i = 1, 2, \dots, m);$$

$$\mathrm{per}(A) = \sum_{i=1}^{m} a_{ij} \mathrm{per}(A(i,j)) \quad (j = 1, 2, \dots, n).$$

We now define the widths and heights of a matrix. In order to simplify the language, we restrict ourselves to (0, 1)-matrices. Let $A = [a_{ij}]$ be a (0, 1)-matrix of size m by n with r_i 1's in row i $(i = 1, 2, \ldots, m)$. We call $R = (r_1, r_2, \ldots, r_m)$ the row sum vector of A. Let α be an integer with $0 \leq \alpha \leq r_i$, $i = 1, 2, \ldots, m$. Consider a subset $J \subseteq \{1, 2, \ldots, n\}$ such that each row sum of the m by $|J|$ submatrix

$$E = A[\cdot, J]$$

of A is at least equal to α. Then the columns of E determine an *α-set of representatives* of A. This terminology comes from the fact that in the configuration of subsets $X_1, X_2, \ldots, X_m$ of $X = \{x_1, x_2, \ldots, x_n\}$ associated with A (see Section 1.1), the set $Z = \{x_j : j \in J\}$ satisfies

$$|Z \cap X_i| \geq \alpha \quad (i = 1, 2, \ldots, m).$$

The *α-width* of A equals the minimal number $\epsilon_\alpha = \epsilon_\alpha(A)$ of columns of A that form an α-set of representatives of A. Clearly, $\epsilon_\alpha \geq |\alpha|$, but we also have

$$0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_r \tag{1.1}$$

where r is the minimal row sum of A. The widths of A are invariant under row and column permutations.

Let $E = A[\cdot, J]$ be a submatrix of A having at least α 1's in each row and suppose that $|J| = \epsilon(\alpha)$. Then E is a *minimal α-width submatrix* of A. Let F be the submatrix of E composed of all rows of E that contain exactly α 1's. Then F cannot be an empty matrix. Moreover, F cannot have a zero column, because otherwise we could delete the corresponding column of E and obtain an m by $\epsilon_\alpha - 1$ submatrix of A with at least α 1's in each row, contradicting the minimality of ϵ_α. The matrix F is called a *critical α-submatrix* of A. Each critical α-submatrix of A contains the same number ϵ_α of columns, but the number of rows need not be the same. The minimal number $\delta_\alpha = \delta_\alpha(A)$ of rows in a critical α-submatrix of A is called the *α-multiplicity* of A. We observe that $\delta_\alpha \geq 1$ and that multiplicities of A are invariant under row and column permutations. Since a critical α-submatrix cannot contain zero columns, we have $\delta_1 \geq \epsilon(1)$.

Let the matrix A have column sum vector $S = (s_1, s_2, \ldots, s_n)$, and let β be an integer with $0 \leq \beta \leq s_j$ $(1 \leq j \leq n)$. By interchanging rows with columns in the above definition, we may define the *β-height* of A to be the minimal number t of rows of A such that the corresponding t by n submatrix of A has at least β 1's in each column. Since the β-height of A equals the β-width of A^{T}, one may restrict attention to widths.

We conclude this section by introducing a parameter that comes from the theory of hypergraphs [1]. Let A be a (0,1)-matrix of size m by n. A *(weak) t-coloring* of A is a partition of its set of column indices into t sets $I_1, I_2, \ldots, I_t$ in such a way that if row i contains more than one 1, then $\{j : a_{ij} = 1\}$ has a nonempty intersection with at least two of the