GROUP THEORY IN PHYSICS

Volume I

J. F. CORNWELL

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GROUP THEORY IN PHYSICS

Volume I

Techniques of Physics

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Techniques of physics find wide application in biology, medicine, engineering and technology generally. This series is devoted to techniques which have found and are finding application. The aim is to clarify the principles of each technique, to emphasize and illustrate the applications and to draw attention to new fields of possible employment.

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Preface

Twenty years or so ago group theory could have been regarded merely as providing a very valuable tool for the elucidation of the symmetry aspects of physical problems, but recent developments, particularly in theoretical high-energy physics, have transformed its role, so that it now occupies a crucial and indispensable position at the centre of the stage. These developments have taken physicists increasingly deeper into the fascinating world of the pure mathematicians, and have led to an ever-growing appreciation of their achievements. That this recognition is in some respects rather belated is to a large extent due to the unfortunate fact that much of modern pure mathematics is written in a style that outsiders find difficult to penetrate. Consequently one of the main objectives of these two volumes (and particularly of the second) has been to help overcome this unnatural barrier, and to present to theoretical physicists and others the relevant mathematical developments in a form that should be easier to comprehend and appreciate.

The main aim of these two volumes has been to provide a thorough and self-contained account both of those parts of group theory that have been found to be most useful and of their major applications to physical problems. The treatment starts with the basic concepts and is carried right through to some of the most significant and recent developments. The areas of physics that appear include atomic physics, electronic energy bands in solids, vibrations in molecules and solids, and the theory of elementary particles. No prior knowledge of group theory is assumed, and for convenience various relevant algebraic concepts are summarized in Appendices A and B.

It need hardly be said that the title that has been chosen for these volumes, "Group Theory in Physics", does not imply that they contain *every* application of group theoretical ideas to physics, nor that the mathematical concepts contained within them are strictly restricted to those of group theory. Some parts of physics, such as nuclear structure theory, have had to be omitted completely. Moreover, even in those areas that have been discussed, a rigorous selection of topics has had to be made. This is particularly so in applications to elementary-particle theory, where literally thousands of papers involving group theoretical techniques have been written. On the other hand, the mathematical coverage goes outside the strict confines of group theory itself, for one is soon led to the study of Lie algebras, which, although related to Lie groups, are often developed by mathematicians as a separate subject.

No apology should be needed for combining such diverse physical applications in the same work, for it is a manifestation of the power of the theory that it has such wide applicability. However, for the benefit of those readers who may wish to concentrate on specific applications, the following list gives the relevant chapters:

(i) molecular vibrations: Chapters 1, 2, and 4 to 7,
(ii) electronic energy bands in solids: Chapters 1, 2, 3 (Section 5 only), 4, 5, 6, 8 and 9,
(iii) lattice vibrations in solids: Chapters 1, 2, and 4 to 9,
(iv) atomic physics: Chapters 1 to 6 and 10 to 12,
(v) elementary particles: Chapters 1 to 6, 10, 11, 12 (except Sections 6 to 8), and 13 to 19.

In the text the treatments of specific cases are frequently given under the heading of "Examples". The format is such that these are clearly distinguished from the main part of the text, the intention being to indicate that the detailed analysis in the Example is not essential for the general understanding of the rest of that section or succeeding sections. Nevertheless, the examples are important for two reasons. Firstly, they give concrete realizations of the concepts that have just been introduced. Secondly, they indicate how the concepts apply to certain physically important groups or algebras, thereby allowing a "parallel" treatment of a number of specific cases. For instance, many of the properties of the groups SU(2) and SU(3) are developed in a series of such Examples.

The *proofs* of theorems have been divided into three categories. First there are those that by virtue of the direct nature of their arguments assist in the appreciation of the theorem. These are included in the main text. Then there are proofs which are worth recording, if only because it is interesting to see to what extent they retain their validity when the conditions of the theorem are changed slightly, as for example when a Lie algebra is generalized to a Lie superalgebra. The arguments involved in these are usually less direct, and so they have been relegated to appendices. Finally there are proofs that are just too long, or involve ideas that have not been developed in these volumes. For these all that is given is a reference or references to works where they may be found.

In the second volume I have chosen to devote much more space to the development of mathematical techniques than to the treatment of specific physical models, largely because the mathematics is likely to be more durable. Consequently not all of the mathematical results that have been derived are actually used explicitly in the models which are discussed in detail. Indeed these models could be regarded as prototypes indicating what can be achieved by this type of argument, rather than as definitive and conclusive statements about the physical world, although the progress with them and the agreement with experimental observations are extremely encouraging. The development of semi-simple Lie algebras follows the classic approach of Cartan, which has the great advantage of being equally applicable to all cases, and treatments that are valid only for restricted types have been largely neglected. A considerable amount of useful data on semi-simple Lie algebras and groups has been presented in the Appendices, some of this having been specially obtained by computer calculation.

I would like to thank Dr A. Cant for his valuable comments on the first drafts of certain chapters, and Miss L. M. McLean, Mrs J. Kubrycht and Mrs N. Pacholek for the excellence of their typing.

J. F. CORNWELL
St Andrews
January 1984

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