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Introduction to categories, homological algebra and sheaf cohomology

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Preface

Categories and functors, it has often been repeated, were introduced thirty years ago by Eilenberg and MacLane [11] to understand and study certain constructions in algebraic topology. It was soon realized that they provided a useful language in which to treat large tracts of mathematics, ranging as far afield as algebraic geometry on the one hand [19] and automata theory [10] on the other. Thus category theory was developed with the specific needs of certain of these fields in mind. Indeed, it is fair to say that many of the most significant contributions came from mathematicians, expert in one or another area, who forged the new theory to their own use. But as the discipline gained momentum, it started generating internal problems of its own, and an ever increasing band of mathematicians who worked on them became known as categorists. In this respect the situation resembles that of group theory. After people had been working with permutation groups, substitution groups, transformation groups for decades, the notion of 'abstract groups' evolved during the third quarter of last century. This general concept rapidly made clear why the older theories had many features in common. In time, however, questions began to be asked in pure group theory which, as everyone knows, were not always easy to answer.

In this book we do not lose sight of the origins of the subject: categories are there to make different topics more transparent by revealing common underlying patterns. This is particularly true of the notion of adjoint functor which is introduced at an early stage and remains a central theme throughout the book. In view of applications, we have also stuck to the traditional description of a category as consisting of objects and morphisms, rather than as just morphisms with certain operations, sometimes favoured by 'pure' categorists.

The material in the first two chapters is mostly standard, but the arrangement perhaps is not. In chapter 1 representable and adjoint functors straightaway take the stage and are used in our treatment of products and limits. The latter owes much to Lambek [23]. Throughout, many examples and exercises should convince the reader that he is doing 'real' mathematics – albeit a rather superficial part of it – and not just building castles in the air. Only in chapter 2 do we get acquainted with monomorphisms and epimorphisms, kernels and cokernels, which feature right at the beginning of most texts, and we gradually introduce more structure in our categories. Thus we pass from additive and exact categories to abelian and Grothendieck categories. We could not resist presenting the pretty juggling of axioms defining an abelian category, mainly following D. Puppe [34]. Our treatment of Grothendieck categories is frankly utilitarian, geared to the needs of homological algebra; our account has benefited from Popescu [33].

Homological algebra also arose out of algebraic topology when its practitioners began to consider homology groups rather than just Betti numbers. Essentially it deals with derived functors; the treatise by Cartan–Eilenberg [9] was followed by those of MacLane [25] and Hilton–Stammbach [20]. Chapter 3 presents the elements of that theory, but without going at all into applications. As opposed to these books, where the theory is set up for modules and it is then remarked as an afterthought that it also carries through for abelian categories, we work with these straightaway as in Mitchell's book [29]. We first present the theory of Yoneda extensions in a given abelian category. From these we build a large new category and by extending a given functor from the original category to the new category we obtain its sequence of satellites in one fell swoop. Thus the Kan extension theorem yields an existence theorem for satellites. This elegant method was suggested by P. Gabriel in his review of Mitchell's book [15]. I am grateful to him for telling me about it back in 1968. The large category involving the Ext's has the additional advantage that the additivity of these functors follows very easily, a fact also noticed by Brinkmann [7] in a similar setting. Our treatment of derived functors is more conventional; it follows the lines laid down by Grothendieck [18]. We do not discuss spectral sequences.

The fourth and final chapter deals with sheaves and their cohomology. This is an important topic in its own right, but also one in which adjoint functors are employed to great advantage. The cohomology of sheaves of modules displays the techniques developed in the chapter on homological algebra. Applications of sheaf cohomology are manifold, in various fields of mathematics. They are not touched upon here; only the elements of the theory are presented. For a more extensive treatment the reader is referred to the monographs [17], [42] and [6].

In some recent books on categories, the author explains in his preface that he intends to write a textbook as well as a work of reference, for students as well as mature mathematicians. This makes four objectives in all, which seems a tall order to fill. So let me state explicitly that this book is meant as a textbook, not a monograph, treatise or work of reference. I have had in mind students rather than mature mathematicians, learners rather than experts. Even so, the wisdom of publishing as a book notes from a course given seven years ago is legitimately open to question. The subject has rapidly developed in the meantime, but I believe that most of the material in this book should still be considered basic. To my mind, the most important developments have been nonabelian homological algebra and the theory of topoi. In both fields, an authoritative treatise still remains to be written; see however [35], [1], and [43] respectively. For both subjects, certain parts of this book form a useful if not absolutely necessary preliminary.

As already mentioned, the book arose out of a course, given at Utrecht University during the first semester of 1968/1969, followed by a seminar. Notes of the course were taken by A. G. van Asch and W. L. J. van der Kallen. In the seminar, S. H. Nienhuys-Cheng and J. W. Nienhuys exposed sheaves and their cohomology. Notes of their lectures were taken by W. H. Hesselink. To all these people the original Dutch notes, put out by Utrecht University in 1970, owe much. Dr Hesselink moreover has helped considerably with the revision of the fourth chapter for the present edition. My colleague Dr C. J. Penning of the University of Amsterdam undertook the translation. However, his contribution has been far greater than the rendering of the text into English. He made many suggestions and revisions and the final form was decided upon during frequent discussions, in which he often managed to boost my flagging morale. Finally I wish to thank J. Lambek for urging me to publish these notes in the first place and for insisting when I remained reluctant; F. Oort for discouraging and P. Gabriel for encouraging the project. All these people share in the merits, if any, of the final result; but only the author is to blame for its shortcomings. And now, gentle reader, bring along an open mind and judge for yourself.

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Contents

1

General concepts

1.1 Categories

1.1.1 Definition A *category* **C** is a system of morphisms and objects. We say that f is a *morphism in* **C** *from the object A to the object B* and write $f: A \to B$ or $A \xrightarrow{f} B$. The following conditions should be satisfied.

(i) For each morphism f in **C** there are unique objects A and B in **C** such that $f: A \to B$.

(ii) For each pair of objects A and B in **C** the class of morphisms f such that $f: A \to B$ is a set (A, B). This set may be empty.

(iii) For all objects A, B and C in **C** there is a mapping (called *composition* or *product*) $(B, C)\times(A, B)\to(A, C)$ which assigns to a pair $\ulcorner g, f\urcorner$, with $g\in(B, C)$ and $f\in(A, B)$, the product $gf\in(A, C)$.

(iv) Existence of identities: for every object A in **C** there is a morphism $1_A: A\to A$ with the property that for every object C in **C** and for every couple of morphisms $f: A\to C$ and $g: C\to A$ we have $f1_A = f$ and $1_A g = g$.

(v) Associativity: for objects A, B, C and D and morphisms $f: A\to B$, $g: B\to C$ and $h: C\to D$ in **C** we have $(hg)f = h(gf)$.

Comments This definition is abstracted from the case that objects are sets and morphisms are mappings. In the abstraction objects are not necessarily sets, nor are morphisms necessarily mappings. The morphisms are the essential ingredients of the theory; the objects are of minor importance.

ad (i). Whether the statement $f: A\to B$ is true or untrue is given together with the category. It is a statement within this category.

ad (ii). Set theory is taken to be known, in particular the difference between class and set. We review this point briefly. A class is not a set if it is bigger than every set. For instance the class of all sets is larger than every set and hence is not a set. If a class K is not larger than a given set X, then K itself is a set. Big sets can be constructed by taking the cartesian product $\prod_{i\in I} A_i$ of sets A_i, where the index set I is also a set. In this book we take a rather naive view of these matters since we are not dealing with foundations. A more careful discussion may be found in [26].

ad (iii). When there is danger of confusion we sometimes specify the category in which morphisms are considered and write $(A, B)_{\mathbf{C}}$. For similar reasons we occasionally write the composition of f and g as $g \circ f$.

ad (iv). The identity morphisms are unique. For the existence of two such identity morphisms 1_A and e_A for an object A implies $1_A = 1_A e_A = e_A$.

Notation Instead of 'A is an object in the category **C**' we write $A \in \mathbf{C}$. This does not therefore have the meaning it has in set theory, since **C** need not be a set.

1.1.2 Examples of categories (*a*) **Sets.** This is the system consisting of sets and mappings. We agree that for each set X there is a unique map going from the void set $\varnothing$ to X. Caution: for $Y \subset Z$ we distinguish between $f\colon X \to Y$ and $g\colon X \to Z$, even when $f(x) = g(x)$ for all elements x of X, in order to comply with axiom (i).

(*b*) **Sets$_*$.** The category of sets with base-points. Objects are nonvoid sets V with a given point $*_V$. Morphisms are mappings that map base-points to base-points.

(*c*) **Top.** This is the category of topological spaces. Objects are topological spaces and morphisms are continuous mappings. **Hausd** is the category of Hausdorff spaces.

(*d*) **Top$_*$.** As above, with base-points.

(*e*) **Gr.** Groups with group-homomorphisms.

(*f*) **Ab.** Abelian groups with group-homomorphisms.

(*g*) $\mathbf{V}_k$. The category of vector spaces over a given field k with linear mappings.

(*h*) **Rg**. This is the category of rings with ring-homomorphisms. Rings are supposed to have an identity element and ring-homomorphisms are supposed to map the identity element to the identity element. The smallest ring consists of only one element.

(*i*) **CRg.** Commutative rings and ring-homomorphisms.

(*j*) $\mathbf{M}_R$. R is a fixed ring. This is the category of right R-modules. The morphisms are R-linear mappings. The modules should be right unitary: $x1 = x$ for all elements x of $M \in \mathbf{M}_R$. Analogously, ${}_R\mathbf{M}$ is the category of left R-modules.

(*k*) **CR-alg.** This is the category of commutative R-algebras with algebra-homomorphisms. The algebras are supposed to have an identity element and the algebra-homomorphisms should transform identity element to identity element.

(l) **TopGr.** Topological groups and continuous group-homomorphisms.

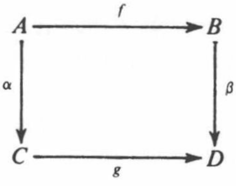
(m) Let I be a preordered class. A category $\mathbf{I}$ is constructed by taking the elements of I as objects. The set of morphisms (i, j) from i to j is empty unless $i \leqslant j$ in which case (i, j) is the set consisting of one element.

(n) Let G be a group. The category $\mathbf{G}$ is the category with single object G and with morphisms all left multiplications.

(o) Let $\mathbf{C}$ be a category. The *dual category* $\mathbf{C}^\circ$ is defined as follows: the objects and morphisms of $\mathbf{C}$ and $\mathbf{C}^\circ$ are the same but the morphisms of $\mathbf{C}^\circ$ run 'in the opposite direction' (arrows are reversed); in other words, for every pair of objects A and B we have $(A, B)_{\mathbf{C}} = (B, A)_{\mathbf{C}^\circ}$. For $f \in (A, B)_{\mathbf{C}}$ we write $f^\circ \in (B, A)_{\mathbf{C}^\circ}$. In the case when $g \in (B, C)_{\mathbf{C}}$, composition in $\mathbf{C}^\circ$ is defined by $f^\circ g^\circ = (gf)^\circ$.

(p) Let $\mathbf{A}$ and $\mathbf{B}$ be categories. The *product category* $\mathbf{A} \times \mathbf{B}$ is defined as follows. Objects are pairs $\ulcorner A, B \urcorner$ of objects with $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Morphisms are pairs $\ulcorner f, g \urcorner$ of morphisms with f a morphism in $\mathbf{A}$ and g in $\mathbf{B}$. The product of any number of categories is defined similarly.

(q) Let $\mathbf{C}$ be a category. One defines a category $\mathbf{C}^2$ in the following way. Objects of $\mathbf{C}^2$ are the morphisms of $\mathbf{C}$. Morphisms of $\mathbf{C}^2$ are certain pairs of morphisms of $\mathbf{C}$. For $f: A \to B$ and $g: C \to D$ in $\mathbf{C}$, the pair $\ulcorner \alpha, \beta \urcorner$ is a morphism from f to g in $\mathbf{C}^2$ if and only if $\alpha: A \to C$ and $\beta: B \to D$ make the following diagram commutative (i.e. $\beta f = g\alpha$):



Note that $\mathbf{C}^2$ is not the same category as $\mathbf{C} \times \mathbf{C}$.

1.1.3 Terminology For $f: A \to B$ in a category $\mathbf{C}$, A is called the *domain* of f and B the *range* of f. We call f an *isomorphism* (notation $f: A \xrightarrow{\sim} B$) and A and B are called *isomorphic* (notation $A \simeq B$) provided there is a morphism $g: B \to A$ in $\mathbf{C}$ such that $fg = 1_B$ and $gf = 1_A$. Given f, such a morphism g is necessarily unique.

A category is called *small* provided the class of objects is a set. In this case the class of all morphisms is also a set since this class equals $\bigcup_{A,B \in \mathbf{C}} (A, B)$ which is a set.

A category is called *concrete* provided its objects are sets endowed with a certain structure which is conserved by morphisms. Examples (*a*) to (*l*) are concrete categories. A precise definition will be given in the next section.

$\mathbf{C}'$ is called a *subcategory* of a category $\mathbf{C}$ provided:

(i) $C' \in \mathbf{C}' \Rightarrow C' \in \mathbf{C}$ for all C';

(ii) $(A, B)_{\mathbf{C}'} \subset (A, B)_{\mathbf{C}}$ for all $A, B \in \mathbf{C}'$;

(iii) $(1_{C'})_{\mathbf{C}'} = (1_{C'})_{\mathbf{C}}$.

$\mathbf{C}'$ is called a *full subcategory* of $\mathbf{C}$ provided it is a subcategory with the stronger condition:

(ii)' $(A, B)_{\mathbf{C}'} = (A, B)_{\mathbf{C}}$ for all A and $B \in \mathbf{C}'$.

For example **Ab** is a full subcategory of **Gr**. The category of all metric spaces with isometries is not a full subcategory of **Top** but **Hausd** is.

1.2 Functors

1.2.1 Definition A *covariant functor* T from a category $\mathbf{C}$ to a category $\mathbf{D}$ is a prescription which assigns to each object $C \in \mathbf{C}$ an object $TC \in \mathbf{D}$ and to each morphism $f \in (A, B)_{\mathbf{C}}$ a morphism $Tf \in (TA, TB)_{\mathbf{D}}$ such that the following conditions are satisfied:

(i) for all $C \in \mathbf{C}$, $T1_C = 1_{TC}$;

(ii) $T(gf) = TgTf$ for all $f \in (A, B)_{\mathbf{C}}$ and all $g \in (B, C)_{\mathbf{C}}$.

Notation $T\colon \mathbf{C} \to \mathbf{D}$ or $\mathbf{C} \xrightarrow{T} \mathbf{D}$.

1.2.2 Examples (*a*) $T\colon$ **Top** $\to$ **Sets**. T is the functor that forgets the topological structure (the *forgetful functor*). Continuous maps between topological spaces are now considered just as maps between the underlying sets.

(*b*) $T\colon$ **Gr** $\to$ $\mathbf{Sets}_*$. Similar to (*a*). In the underlying set TG for $G \in$ **Gr**, take the identity element as the base-point $*$.

(*c*) Let G and H be groups and let $f \in (G, H)_{\mathbf{Gr}}$. Consider the categories $\mathbf{G}$ and $\mathbf{H}$ as described in 1.1.2(*n*). Then one may define $T\colon \mathbf{G} \to \mathbf{H}$ by $TG = H$ and $T(\lambda_a) = \lambda_{f(a)}$ (λ_a: left multiplication by a in G).

(d) $T\colon$ **Gr** $\to$ **Gr**. For $G \in$ **Gr** let $TG = [G, G]$ (commutator subgroup of G) and for $f \in (G, H)_{\mathbf{Gr}}$ let Tf be the restriction of f to $[G, G]$.

(*e*) $T\colon$ **Gr** $\to$ **Ab**. For $G \in$ **Gr** let $TG = G/[G, G]$ and for $f \in (G, H)_{\mathbf{Gr}}$ let Tf be defined by $Tf(a[G, G]) = f(a)[H, H]$, $a \in G$.

(*f*) $T\colon \mathbf{Top}_* \to$ **Gr**. For $(X, *) \in \mathbf{Top}_*$, $T(X, *) = \pi(X, *)$ (fundamental group of X with respect to the base-point $*$). See [39, 1.8].

(*g*) $T\colon$ **Top** $\to$ **Ab**. $T = H_n$ (n^{th}-singular homology functor). See [39, 4.4].

(*h*) Let $R \in \mathbf{Rg}$ and $N \in \mathbf{M}_R$. $T: {}_R\mathbf{M} \to \mathbf{Ab}$ is defined by $TM = N \otimes_R M$ and for $f: M \to L$ in ${}_R\mathbf{M}$ by $Tf = 1 \otimes f: N \otimes_R M \to N \otimes_R L$. If R is commutative there is no distinction between left and right modules. In that case $N \otimes_R M$ is an R-module and one may consider T as a functor from ${}_R\mathbf{M}$ to ${}_R\mathbf{M}$.

1.2.3 Terminology A functor $T: \mathbf{C} \to \mathbf{D}$ is called *faithful* provided $Tf = Tg$ implies $f = g$; i.e. for all $A, B \in \mathbf{C}$ the mapping $T: (A, B)_\mathbf{C} \to (TA, TB)_\mathbf{D}$ is injective. If all these mappings are surjective, the functor is called *full*. T is called an *embedding* provided T is faithful and $TA = TB$ implies $A = B$. A category $\mathbf{C}$ is called *concrete* provided there is a faithful functor $T: \mathbf{C} \to \mathbf{Sets}$. This makes precise the description of a concrete category in 1.1.3.

Let $\mathbf{C}$ be any category and $C \in \mathbf{C}$. Consider the covariant functor $h_C: \mathbf{C} \to \mathbf{Sets}$ defined by $h_C A = (C, A)$ for $A \in \mathbf{C}$ and $h_C f(u) = fu$ for $f \in (A, B)_\mathbf{C}$, $u \in (C, A)_\mathbf{C}$. This functor is basic in category theory, since it describes the composition of morphisms in the given category.

1.2.4 Definition A *contravariant functor* T from a category $\mathbf{C}$ to a category $\mathbf{D}$ is defined as in 1.2.1 except that now $Tf \in (TB, TA)_\mathbf{D}$ and condition (ii) reads $T(gf) = Tf \circ Tg$.

An important example is the contravariant functor (for $C \in \mathbf{C}$) $h^C: \mathbf{C} \to \mathbf{Sets}$ defined by $h^C A = (A, C)$ for $A \in \mathbf{C}$ and $h^C f(u) = uf$ for $f \in (A, B)_\mathbf{C}$, $u \in (B, C)_\mathbf{C}$.

The contravariant functor $^\circ: \mathbf{C} \to \mathbf{C}^\circ$ introduced in 1.1.2 (*o*) will be denoted by Δ (the dualizing functor). Thus Δ is defined by $\Delta C = C$ and for $f: C \to C'$ by putting $\Delta f = f^\circ: C' \to C$. For any contravariant functor $T: \mathbf{C} \to \mathbf{D}$ one can consider the compositions $T\Delta: \mathbf{C}^\circ \to \mathbf{D}$ and $\Delta T: \mathbf{C} \to \mathbf{D}^\circ$ which are covariant. In this way we often identify a contravariant functor $T: \mathbf{C} \to \mathbf{D}$ with its covariant counterpart $T\Delta: \mathbf{C}^\circ \to \mathbf{D}$. When this does not give rise to confusion we sometimes drop the Δ.

A contravariant functor T is called full, faithful or an embedding functor provided $T\Delta$ is such.

Some more examples of contravariant functors are:

(*a*) $H^n: \mathbf{Top} \to \mathbf{Ab}$ with H^n the n^{th}-cohomology functor. See [28, ch. 5], [39, ch. 6].

(*b*) Spec: $\mathbf{CRg} \to \mathbf{Top}$ (the spectrum of a commutative ring, see [19]).

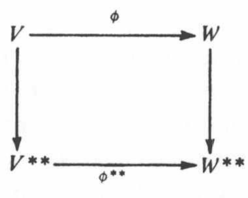
'Functor' on its own usually means a convariant one.

1.2.5 Multifunctors One can also consider functors of more than one variable. Such a functor may be covariant in all variables, contravariant in all variables, or covariant in some and contravariant in the other variables. The reader should write out the formulas for this situation for himself. As a typical example we give the following special case.

Let **C** be any category. For any two objects A and B in **C** denote $(A, B)_{\mathbf{C}}$ by $H\ulcorner A, B\urcorner$. Then H is a *bifunctor* (functor of two variables), contravariant in the first and covariant in the second variable. We consider this as a covariant functor from $\mathbf{C}^{\circ}\times\mathbf{C}$ to **Sets**. Explicitly, if $f^{\circ}: A'\to A$ (i.e. $f: A\to A'$) and $g: B\to B'$, then $H\ulcorner f, g\urcorner: H\ulcorner A, B\urcorner\to H\ulcorner A', B'\urcorner$ is given by $H\ulcorner f, g\urcorner(u)=g\circ u\circ f^{\circ}$ for $u\in H\ulcorner A, B\urcorner$. $H\ulcorner f, g\urcorner$ is also denoted by $(f, g)_{\mathbf{C}}$. We often denote H by $(-, -)_{\mathbf{C}}$.

1.3 Morphisms of functors

1.3.1 Example Let V be a vector space over a field k and V^{**} its double dual which in our language would be denoted by $((V, k)_{\mathbf{V}_k}, k)_{\mathbf{V}_k}$. There is a particular linear mapping from V to V^{**} that has some remarkable properties which we shall describe now. First let the linear mapping $\hat{}: V\to V^{**}$ be defined by $\hat{v}(f)=f(v)$ for $v\in V$ and $f\in V^*$ (the dual of V). Now let $\phi: V\to W$ be a linear map. Let $\phi^{**}: V^{**}\to W^{**}$ be the corresponding map between the double duals. The following diagram is then commutative:



i.e. $\hat{}\circ\phi=\phi^{**}\circ\hat{}$, as the reader may easily check.

1.3.2 Definition Let **C** and **D** be categories and S and T functors from **C** to **D**:

$$\mathbf{C}\ \underset{T}{\overset{S}{\rightrightarrows}}\ \mathbf{D}$$

A *morphism* η *from* S *to* T is a class of morphisms $\eta(C)$ in **D**, indexed by the objects C of **C**, such that

(i) $\eta(C): SC\to TC$ in **D** for all $C\in\mathbf{C}$;

(ii) for every $f: A\to B$ in **C** the following diagram is commutative:

$$\begin{array}{ccc} SA & \xrightarrow{Sf} & SB \\ {\scriptstyle \eta(A)}\downarrow & & \downarrow{\scriptstyle \eta(B)} \\ TA & \xrightarrow[Tf]{} & TB \end{array}$$

Example 1.3.1 shows a morphism ^ from the identity functor $I: \mathbf{V}_k \to \mathbf{V}_k$ to the 'double dual' functor $**: \mathbf{V}_k \to \mathbf{V}_k$. If one tries to define such a morphism of functors between the identity functor and the 'single dual' functor * one runs into the difficulty that there is no candidate to replace ^ for this situation. Even if one restricts to the category of finite dimensional vector spaces over k, so that one knows that V and V^* are isomorphic, these isomorphisms depend on the choice of bases and we cannot choose linear maps $\eta(V): V \to V^*$ such that condition (ii) of definition 1.3.2 is satisfied. It is this difference between the mappings $\hat{}(V): V \to V^{**}$ and $\eta(V): V \to V^*$ that motivates the name of 'natural transformation' for the first. It is in this sense that the mysterious word 'canonical' is mostly used. Thus a morphism η of functors is also called a *natural transformation of functors*. As such, this important notion was introduced in the paper of Eilenberg and MacLane [11], which is at the origin of category theory. Indeed, the example of vector spaces and their double duals is theirs.

If for all the objects C of the category $\mathbf{C}$ the morphisms $\eta(C)$ are isomorphisms, the morphism η of functors is called a *natural equivalence* (see 1.3.4).

1.3.3 Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Consider the 'category' $(\mathbf{C}, \mathbf{D})$ whose objects are the covariant functors from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are the functor morphisms defined in 1.3.2. Is $(\mathbf{C}, \mathbf{D})$ a category? It is easily verified that with the obvious composition of morphisms of functors the axioms for a category are satisfied except perhaps axiom (ii). If the category $\mathbf{C}$ is small this axiom is satisfied since for functor morphisms $S, T: \mathbf{C} \to \mathbf{D}$ we have

$$(S, T) \in \prod_{C \in \mathbf{C}} (S(C), T(C))$$

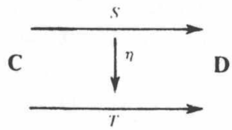
and since the right hand side is a set; see 1.1.1 comment (ii).

If the category $\mathbf{C}$ is not small one sometimes may get around this difficulty as will be seen later (1.9.8 and following).

Although not always justified we still use notation and terminology for $(\mathbf{C}, \mathbf{D})$ as if it were a category, except, of course, in those cases where the missing axiom is essential. Thus we will not use, for instance, the notation h^T for $T \in (\mathbf{C}, \mathbf{D})$ since this would mean $h^T: (\mathbf{C}, \mathbf{D}) \to \mathbf{Sets}$.

Categories allow one to define functors between them. Natural transformations between functors then occur as the morphisms in a 'category' where objects are the functors between two given categories. This 'closure of category theory within itself' is of fundamental importance, both from a foundational point of view, and in the more sophisticated branches of the theory. Brashly, one sometimes speaks of the 'category' **Cat**: its objects are all categories (or possibly all small categories) while $(\mathbf{C}, \mathbf{D})_{\mathbf{Cat}}$ consists of all functors from **C** to **D**.

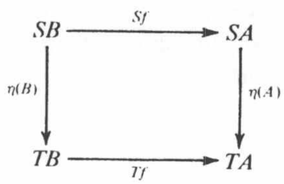
1.3.4 Remark If a morphism η of functors



is such that for every $C \in \mathbf{C}$ the morphism $\eta(C)$ is an isomorphism in the category **D**, the reader may easily establish the fact that then the morphism η has the usual properties required for an isomorphism in the 'category' $(\mathbf{C}, \mathbf{D})$, and vice versa. Therefore a natural equivalence $\eta: S \to T$ is also called an isomorphism from S to T. The functors S and T are called *naturally equivalent* or *isomorphic*. We denote this by $S \simeq T$.

1.3.5 Definition Let S and T be contravariant functors from **C** to **D**. A morphism η from S to T assigns to each $C \in \mathbf{C}$ a morphism $\eta(C)$ such that

(i) $\eta(C): SC \to TC$ for all $C \in \mathbf{C}$;
(ii) for $f: A \to B$ in **C** the following diagram is commutative:



In other words, η is a morphism between the covariant functors $S\Delta$ and $T\Delta$ from $\mathbf{C}^\circ$ to **D**.

1.3.6 Some more examples (*a*) There is a completely analogous morphism of functors $\hat{}: I \to {**}$ from the identity functor to the