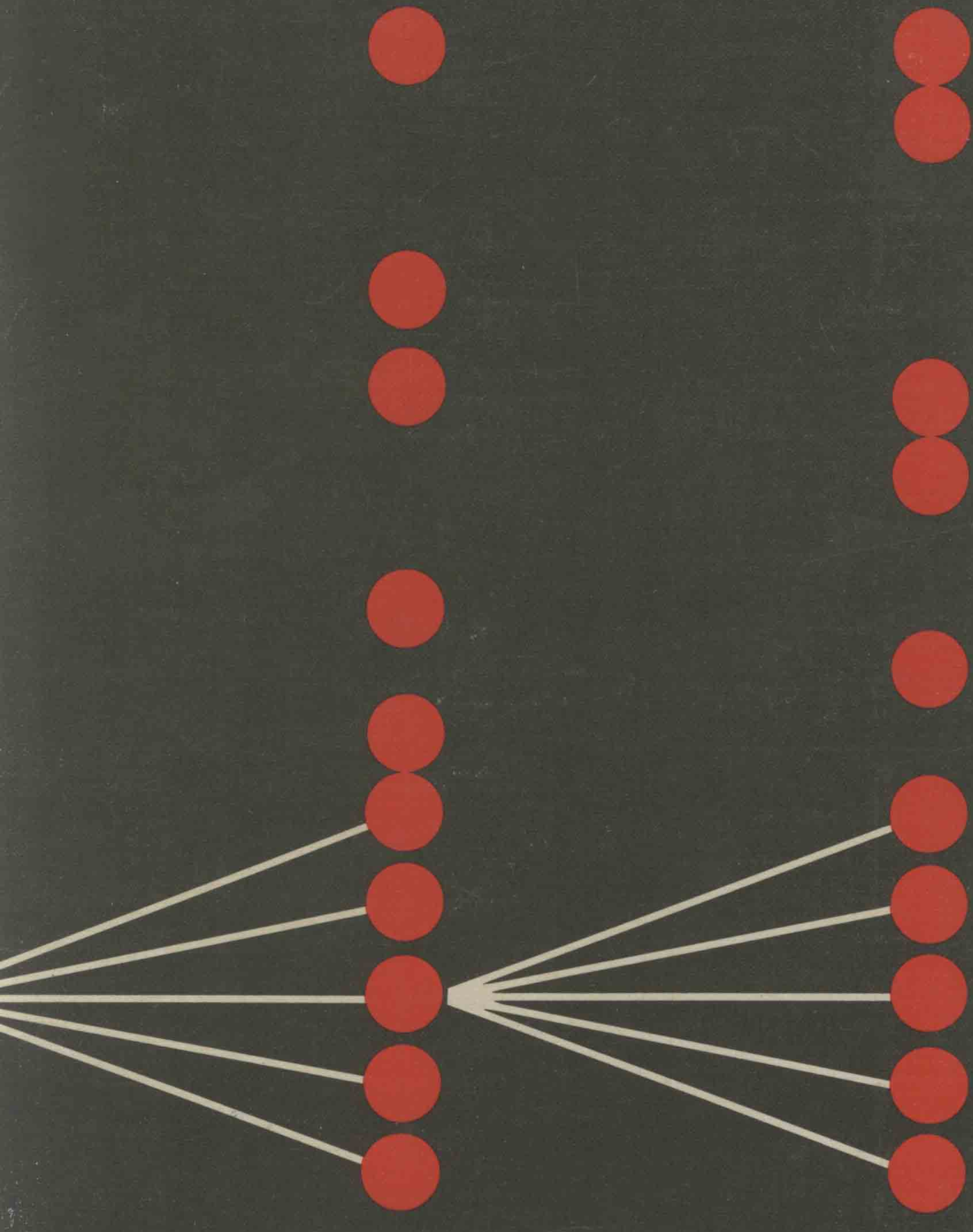
A Survey of Finite Mathematics

Marcus



a survey

Marvin Marcus
The University of California
at Santa Barbara

# of finite mathematics

Houghton Mifflin Company · Boston
New York · Atlanta
Geneva, Illinois · Dallas · Palo Alto

TO REBECCA ELIZABETH

Printed in the USA

Preface

The primary purpose of this book is for use as a text in courses usually entitled *Finite Mathematics* that have come into existence over the last few years in many colleges and universities. Such courses are normally one quarter or one semester in length and are intended for students from the social and biological sciences, business administration, and liberal arts (and even mathematics!). The first chapter of *A Survey* is specifically designed for this audience in both level and content. It is divided into seven sections covering basic mathematical concepts: logic; sets; functions; induction and combinatorics; partitions; probability; stochastic processes. This material can easily be covered in 30 lecture hours and constitutes the first third of the book.

The second two thirds of the book is devoted to the basic ideas from linear algebra and the theory of convex sets. The material from these disciplines constitute the fundamental mathematical tools used in the applications to linear programming, game theory, and Markov chains which appear in the third chapter.

The first section of Chapter 2 is a completely self-contained introduction to vectors and matrices motivated by simple examples from the social and biological sciences. The next five sections are somewhat unique for an elementary book. In these, the student is introduced to the elementary notions of combinatorial matrix theory: incidence matrices, systems of distinct representatives, and stochastic matrices. These topics have far-reaching applications in such diverse fields as communication networks, sociometric relations, operations research, and statistics. The last two sections of Chapter 2 are devoted to a development of the theory and applications of systems of linear equations.

The third and final chapter of the book, entitled *Convexity*, immediately starts with examples of simple linear programming problems. The basic geometry of convex sets, including the theory of maxima and minima for linear functions, appears in the first three sections of this chapter. The remainder of the chapter is devoted to game theory and Markov chains. The treatment of game theory uses techniques from earlier material on convex sets and matrix theory to solve matrix games. The section on Markov chains contains a complete treatment of the elementary aspects of this subject and includes numerous applications.

There are approximately 150 worked examples in the text. These cover a wide range of applications and form an integral part of the material. Most of these are routine, but a few require some thought. Each of the 19 sections of the book ends with a true-false quiz and a set of exercises. Altogether there are over 1200 exercises in the book. Many of the more difficult exercises are accompanied by "hints" for solutions that in some cases constitute complete analyses. Those exercises that are somewhat more difficult are marked with an asterisk. Exercises which are marked with a dagger contain results or definitions that are used elsewhere in the text. These exercises should at least be read and hopefully worked. The quizzes are intended to remind the reader of the essential points covered in the section. Experience at the author's institution indicates that the quizzes are highly effective teaching aids.

In general, then, this is a book on "applied mathematics". In its entirety it is suitable for a two-quarter or one-semester course, or a three quarter or two semester course if the pace is more leisurely. It is our belief that the material is appropriate and important for mathematics majors as well as students from other disciplines. The first chapter of *A Survey* has been used in manuscript form at the University of California, Santa Barbara for a freshman course for students from other departments. The material in Chapters 2 and 3 has been used for a course on discrete applications of matrix theory in a conference for college teachers sponsored on this campus by the National Science Foundation.

I would like to express my thanks to Miss Susan Katz, Miss Barbara Smith, and Mrs. Nancy Stuart for their invaluable assistance in the preparation of this manuscript. I am also very grateful to Mrs. Wanda Michalenko and Mrs. Delores Brannon for their extremely professional jobs in typing and assembling this manuscript. Professor B. N. Moyls of the University of British Columbia acted as a referee on this manuscript. His remarks and suggestions were of inestimable value.

Marvin Marcus

Numbering System

Each of the chapters is divided into sections. Thus the fifth section of Chapter 1 is Section 1.5. Definitions, theorems, and examples are numbered separately within each section. Thus Theorem 3.2 is the second theorem in the third section of the chapter in which it appears. Reference to a theorem (definition, example) is by its number alone when the reference is in the same chapter. If necessary, we give the section number or the chapter number in which the item appears.

Asterisks on exercises indicate that the problem is somewhat difficult. These exercises are usually accompanied by hints, or their solutions appear in the Answers. A limited number of exercises are marked with a dagger †. These exercises contain results that are used within the text and should be carefully read and worked out. This is particularly the case for Exercises 7–18 in Section 2.6 in which the properties of linear independence are developed. (Detailed solutions to this sequence of exercises are included in the Answers.)

Contents

Chapter 3 Convexity

fundamentals

chapter 1

1.1

Truth Tables and Applications

It is appropriate to begin a study of mathematics with a brief discussion of logic. In thinking about any organized body of information we should have some idea of the mechanical rules used in manipulating this information. This comment applies with equal validity to everyday situations. We are all familiar with rather obviously fallacious arguments. For example, any Communist advocates armed revolution; Mr. X advocates armed revolution; therefore Mr. X is a Communist. This argument is, of course, incorrect, for, we can all think of members of political groups who advocate armed revolution and who are, in fact, quite antagonistic to the Communist doctrine.

In general, the study of elementary logic has several purposes. Probably the most important of these purposes is to give precise meaning to such words and phrases as "and", "or", "not", "if . . . then", and "if and only if", which occur throughout any mathematical theory. Second, we want to make somewhat more precise the laws of inference and deduction that are constantly used in constructing a mathematical theory, i.e., we want to have at least a rudimentary idea of what constitutes a coherent mathematical argument. However, even an intimate knowledge of formal logic is no guarantee against error in a mathematical argument; such knowledge simply diminishes the chances of making certain "obvious" mistakes in reasoning. A third and more utilitarian justification for studying logic is its use in such practical applications as the analysis of switching networks.

The reader will recall that in elementary plane geometry one starts with certain primitive or "undefined" items such as "points", "lines", etc. There follows a set of "axioms" and "postulates" governing the relationships of these items, e.g., "two distinct points determine a line." One does not attempt to define "points", "lines", etc. in terms of simpler notions, nor to prove the axioms and postulates. These constitute the starting point of Euclidean geometry. In general, the basic ingredients of a mathematical theory are the following:

A. a set of undefined objects;

B. a certain set of statements or axioms relating these undefined objects;

C. a sequence of statements or theorems which concerns the undefined objects and which are obtained by the rules of logic.

In the development of a mathematical theory, we put together statements with *connectives* to obtain new statements. For example, if p and q represent statements, we may build up compound statements "p and q," "p or q," "not p," etc. We shall now introduce the connectives used in standard logical systems and develop the symbolism used to designate them. The first connective is the word "and"; the symbol used to denote this word is

(1) $$\wedge .$$

The result of putting two statements together with the word "and" is referred to as a *conjunction*.

The second connective is the word "or," which is denoted by the symbol

(2) $$\vee .$$

Joining two statements by the word "or" results in a statement referred to as a *disjunction*.

The word "not" is symbolized by

(3) $$\sim$$

and inserting "not" at the beginning of a statement results in a statement called a *negation*.

The fourth basic symbol stands for "if . . . then" and the symbol is an arrow,

(4) $$\rightarrow.$$

An "if . . . then" statement is usually called an *implication.*

Thus, if p and q are statements, they may be connected symbolically by

$$p \wedge q,$$

read "p and q";

$$p \vee q,$$

read "p or q";

$$\sim p,$$

read "not p"; and finally

$$p \rightarrow q,$$

read "if p then q" or "p implies q."

We shall assume that to any meaningful statement it is possible to assign a *truth value,* namely true (T) or false (F). Observe that this is indeed an assumption. For, consider the statement, "The number of electrons in the universe exceeds 10^{1000}." Although this statement seems to make sense, it is not likely that we can decide whether it is true or false.

We can give meaning to the connectives described above by assigning truth values to each of the four statements $p \wedge q$, $p \vee q$, $\sim p$, and $p \rightarrow q$ as the truth values for p and q vary individually. We do this in a convenient tabular form known as a *truth table.* In each of the following tables, we think of p and q as each standing for entire sentences: p and q can each have one of two truth values, T or F. The last column of each truth table tells us the resulting truth value for the appropriate compound statement formed from p and q.

(5) $\wedge$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(6) $\vee$:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(7) $\sim$:

p	$\sim p$
T	F
F	T

(8) $\rightarrow$:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Thus $p \wedge q$ is false unless both p and q are separately true; $p \vee q$ is true unless each of p and q is individually false. For example, if p is the statement "$2 + 2 = 5$" and q is the statement "dogs are animals," then the statement "$p \vee q$" is true. Table (7) for negation is self-explanatory and reasonable; for if p is true then $\sim p$ is false, and conversely, if p is false then $\sim p$ is true. From Table (8) we see that the statement $p \rightarrow q$ will be true unless p is true and q is false. In other words, we never want a true statement to imply a false one. The fact that $p \rightarrow q$ is true when p is false, regardless of the truth value of q, may require some additional explanation, for this is not the way implication is used in ordinary language. In conversation one usually has some causal connection in mind between p and q in an implication. Thus the statement, "If men are dogs, then women are cats" is meaningful, but is not one that would often be said. Of course, it is false that men are dogs and equally false that women are cats. Nevertheless we want every meaningful statement to have a definite truth value, either T or F, and Table (8) stipulates that "If men are dogs then women are cats" has truth value T. Another way of saying this is that Table (8) actually defines the connective $\rightarrow$. As another example, consider the implication $p \rightarrow q$ where p is the statement "n is a number greater than 17 and less than 3" and q is the statement "$n = 5$." Even though p is false we want the implication to be true, and this can be justified by observing that there is no number n (whether it is 5 or not) which is greater than 17 and less than 3. Although this may seem a little silly, it is important that establishing the truth of $p \rightarrow q$ not carry with it the burden of exhibiting a formal connection between p and q.

Using these elementary connectives we can formulate compound statements that are quite complicated.

Example 1.1 Construct a truth table for the statement $p \rightarrow (p \vee \sim p)$. In other words, we want to assign a truth value to the preceding statement for each of the two possible truth values for p.

p	$\sim p$	$p \vee \sim p$	$p \rightarrow (p \vee \sim p)$
T	F	T	T
F	T	T	T

We filled in the second column by using Table (7), the third column using Table (6) and the fourth column using Table (8). Thus, when the truth value of p is T then the truth value of $p \vee \sim p$ is T (Table (6), row 2), and hence the truth value of the compound statement $p \rightarrow (p \vee \sim p)$ is T by the first row of (8). Similarly, when the truth value of p is F then the truth value of $p \vee \sim p$ is T from the third row of Table (6), and the truth value of $p \rightarrow (p \vee \sim p)$ is T by the third row of Table (8).

A compound statement is said to be *valid* or a *tautology* if its truth value is T regardless of the truth values of its component statements. Thus from Example 1.1 we see that the implication $p \rightarrow (p \vee \sim p)$ is a tautology.

Another useful connective can be defined as follows: the compound statement

(9) $$(p \rightarrow q) \wedge (q \rightarrow p)$$

will be abbreviated to

(10) $$p \leftrightarrow q.$$

The formula (10) is read "p if and only if q." The statements p and q in (10) are sometimes said to be *equivalent*.

Example 1.2 Construct a truth table for the statement $p \leftrightarrow q$.

(11)

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The third and fourth columns of Table (11) are read from Table (8), e.g., in column 4, row 3 of (11), q has truth value T, p has truth value F, and from the second row of (8), we see that $q \to p$ has truth value F. The fifth column of (11) is obtained by joining the third and fourth columns with the connective $\wedge$ and using Table (5), e.g., when $p \to q$ has truth value T and $q \to p$ has truth value F, then $p \leftrightarrow q$ has truth value F, as one sees from the second row of (5).

Example 1.3 Show that the following compound statement is valid:

$$f\colon\ [(\sim p \vee q) \wedge (p \vee \sim q)] \leftrightarrow (p \leftrightarrow q).$$

Consider the table

p	q	$\sim p \vee q$	$p \vee \sim q$	$3 \wedge 4$	$p \leftrightarrow q$	f
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

where in column 5 we have written $3 \wedge 4$ to denote the conjunction of the statements in columns 3 and 4.

Example 1.4 Show that the following compound statement is valid:

$$f\colon\ (p \wedge (p \to q)) \to q.$$

p	q	$p \to q$	$p \wedge (p \to q)$	f
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The fundamental assumption that we shall make about valid compound statements or tautologies is that *they represent correct arguments in any mathematical system.* In other words, it will be assumed that tautologies represent arguments which are acceptable in establishing the theorems in a mathematical theory. To illustrate this, consider the following kind of reasoning: "If p is the case, and whenever p is the case it follows that q is the case, then q must hold." Put more succinctly: "If p, and p implies q, then q." If we state this in logical symbolism, we obtain the compound statement

$$f\colon\ (p \wedge (p \to q)) \to q.$$

But we saw in Example 1.4 that f is a tautology, i.e., that f is "true," or has truth value T, whatever the truth values of p and q may be. The fact that $(p \wedge (p \to q)) \to q$ is a tautology is usually referred to as the *law of detachment* or, in somewhat more rarefied terms, "modus ponens." What we have actually done is set up the truth tables for implication and conjunction in such a way that they yield the law of detachment as a tautology.

As another example, consider the following statement: "If p always implies q and q fails to be the case, then p cannot hold." This is a very familiar and acceptable form of argument used not only in mathematics but in everyday life. It is known as an *indirect proof* or, in Latin, "reductio ad absurdum." For example, suppose it is the case that whenever it is raining I invariably carry my umbrella, and suppose I am not carrying my umbrella. Knowing these two facts, you can conclude that it is not raining. The symbolic statement of the method of indirect proof takes the following form:

$$f\colon\ ((p \to q) \wedge \sim q) \to \sim p.$$

Consider the truth table for f.

p	q	$p \to q$	$\sim q$	$\sim p$	$(p \to q) \wedge \sim q$	f
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Thus the statement f is a tautology and, by our fundamental agreement, represents a correct argument.

A somewhat more subtle argument mentioned earlier is widely used: "If x is a Communist, then x advocates armed revolution. Moreover, x advocates armed revolution. It follows that x is a Communist." This argument is of course rubbish, since x could equally well be a Minute Man. The argument has the following form: "If p implies q, and q, then p." Denote this statement by f and consider the truth table:

$$f\colon\ ((p \to q) \wedge q) \to p.$$