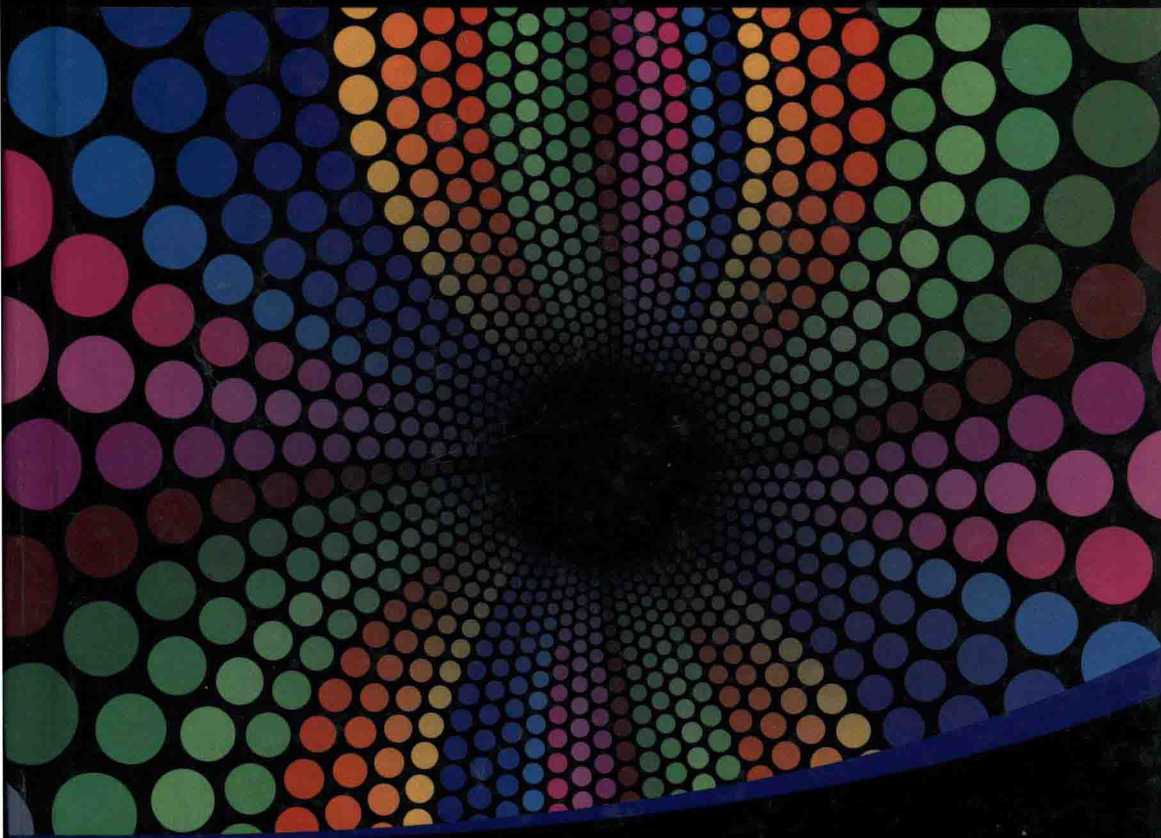
Yi-Bing Shen · Zhongmin Shen

Introduction to Modern Finsler Geometry

现代芬斯勒几何初步



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Introduction to Modern Finsler Geometry

This comprehensive book is an introduction to the basics of Finsler geometry with recent developments in its area. It includes local geometry as well as global geometry of Finsler manifolds.

In Part I, the authors discuss differential manifolds, Finsler metrics, the Chern connection, Riemannian and non-Riemannian quantities. Part II is written for readers who would like to further their studies in Finsler geometry. It covers projective transformations, comparison theorems, fundamental group, minimal immersions, harmonic maps, Einstein metrics, conformal transformations, amongst other related topics. The authors made great efforts to ensure that the contents are accessible to senior undergraduate students, graduate students, mathematicians and scientists.

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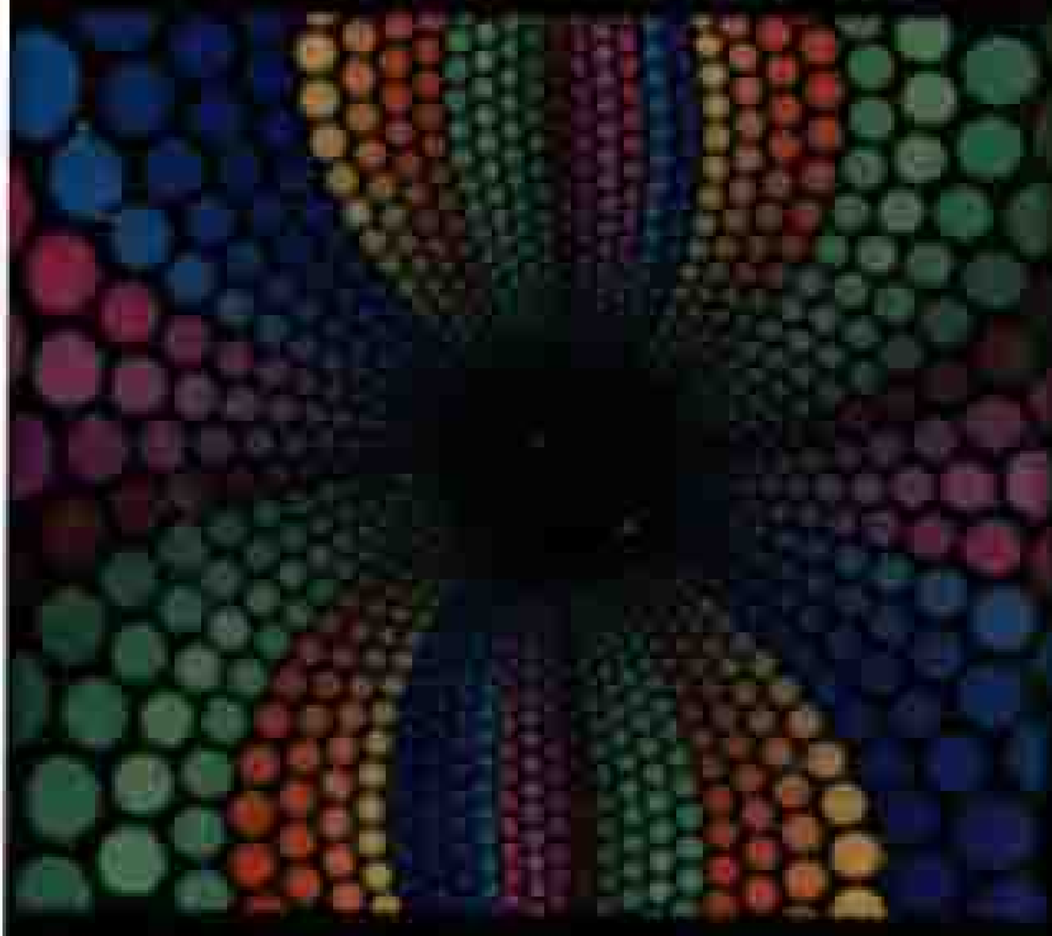

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Preface

"Finsler geometry is just Riemannian geometry without quadratic restriction", commented by the past famous geometer, S. S. Chern. In fact, early in 1854, B. Riemann had introduced the concept of Finsler geometry in his ground-breaking Habilitationsvortrag. He had seen the difference between metrics of quadratic type (i.e. Riemannian metrics) and those in the general case. No essential development was made until 1918 when P. Finsler studied the geometry of curves and surfaces in the general case. Therefore, more precisely, we should call this subject Riemann-Finsler geometry.

Since more than twenty years ago, substantial progress has been made in Finsler geometry, especially in global Finsler geometry, so that we have seen a completely new outlook. Informally speaking, Riemannian geometry studies spaces with only black and white colors, while Finsler geometry studies a colorful world. The methods and ideas used in Finsler geometry not only are closely related to other mathematical branches such as differential equations, Lie groups, algebra, topology, nonlinear analysis, etc., but also have more and more applications to mathematical physics, theoretical physics, mathematical biology, control theory, informatics, etc. Therefore, not just in theory but also in application, Finsler geometry has shown its strong vitality and great value.

In order to meet the need of education for senior undergraduate and graduate students, under the influence of books [10, 103], we wrote this textbook, based on many years of teaching experience. The whole book is divided into 11 chapters: in the first five chapters, we discuss differential manifolds, Finsler metrics, the Chern connection, Riemannian and non-Riemannian quantities. The rest is written for further studies. This second part covers projective transformations, comparison theorems, fundamental group, minimal immersions, harmonic maps, Einstein metrics, conformal transformations and conformal vector fields, the Finsler Laplacian and its first eigenvalue, etc. At the end of every chapter there are some exercises, which are important complements for the contents. The final Appendix is to provide Maple programs on the computations of geometric quantities in Finsler geometry.

With the main tool of tensor analysis, we systematically introduce the basic concepts and methods in Finsler geometry, and we do our best to include the classical theory as well as the newest developments, so that readers can do research independently after studying this book. This book may be used as a selective textbook for senior undergraduate students and a regular textbook for graduate students. It can also be used as a reference book for mathematical physics, theoretical physics, control theory, etc. We believe that this book is of positive significance as an addition and improvement to current textbooks in colleges.

We would like to take this opportunity to thank the National Natural Science Foundation of China (No. 11171297), the Center of Mathematical Sciences and the Department of Mathematics at Zhejiang University, and our many students who contributed to this book.

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Yi-Bing Shen
Zhongmin Shen
March 2015

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PART I

Foundations

Chapter 1

Differentiable Manifolds

1.1 Differentiable manifolds

1.1.1 Differentiable manifolds

Definition 1.1. An n-dimensional *differentiable manifold* is a Hausdorff topological space M and a family of diffeomorphisms $\phi_\alpha : U_\alpha \subset M \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ such that the following conditions are satisfied:

(i) $\{U_\alpha\}$ is an open covering of M, i.e., U_α is open and $\bigcup_\alpha U_\alpha = M$;

(ii) If $U_\alpha \cap U_\beta = V \neq \emptyset$, then $\phi_\alpha(V)$ and $\phi_\beta(V)$ are open sets in $\mathbb{R}^n$, and $\phi_\alpha \circ \phi_\beta^{-1}|_{\phi_\beta(V)}$ is a diffeomorphism;

(iii) The family $\{(U_\alpha, \phi_\alpha)\}$ is maximal relative to (i) and (ii).

For a point $p \in U_\alpha$, (U_α, ϕ_α) is called a *coordinate neighborhood* of p, and the coordinate of $\phi_\alpha(q) \in \phi_\alpha(U_\alpha)$, $\forall q \in U_\alpha$, in $\mathbb{R}^n$ can be viewed as the coordinate of $q \in U_\alpha \subset M$. A family $\{(U_\alpha, \phi_\alpha)\}$ satisfying (i) and (ii) is called a *differentiable structure* on M. The condition (iii) is included for purely technical reasons.

1.1.2 Examples of differentiable manifolds

Example 1.1. The Euclidean space $\mathbb{R}^n$.

Example 1.2. The unit sphere in $\mathbb{R}^{n+1}$

$$\mathbb{S}^n = \left\{ (y^1, \cdots, y^{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{\alpha=1}^{n+1} (y^\alpha)^2 = 1 \right\} \subset \mathbb{R}^{n+1}.$$

Take the topology of $\mathbb{S}^n$ as the sub-topology in $\mathbb{R}^{n+1}$, i.e., $U \subset \mathbb{S}^n$ is open if and only if there is an open $\widetilde{U} \subset \mathbb{R}^{n+1}$ such that $U = \widetilde{U} \cap \mathbb{S}^n$. Thus, $\mathbb{S}^n$ is a Hausdorff topological space. We are going to introduce a differentiable structure on $\mathbb{S}^n$.

For all $1 \le \alpha \le n+1$, let

$$\widetilde{U}_\alpha^+ = \{(y^1, \cdots, y^{n+1}) | y^\alpha > 0\}, \quad \widetilde{U}_\alpha^- = \{(y^1, \cdots, y^{n+1}) | y^\alpha < 0\}.$$

$\widetilde{U}_\alpha^\pm$ are two open sets in $\mathbb{R}^{n+1}$ separated by the hyperplane $y^\alpha = 0$. Then the family $\{U_\alpha^\pm = \widetilde{U}_\alpha^\pm \cap \mathbb{S}^n\}$ covers $\mathbb{S}^n$. Take the orthogonal projection $\phi_\alpha^\pm : U_\alpha^\pm \to \mathbb{R}^n$

$$\phi_\alpha^\pm(y^1, \cdots, y^{n+1}) = (y^1, \cdots, \widehat{y^\alpha}, \cdots, y^{n+1}),$$

where $\widehat{y^\alpha}$ means the corresponding term is omitted. It is easy to see that $\phi_\alpha^\pm$ are diffeomorphisms from $U_\alpha^\pm$ to the open set $W_\alpha = \{(y^1, \cdots, \widehat{y^\alpha}, \cdots, y^{n+1}) \in \mathbb{R}^n | \sum_{\beta \ne \alpha}(y^\beta)^2 < 1\}$. Moreover, coordinate transformations are smooth. In fact, for example, the transformation $\phi_2^- \circ (\phi_1^+)^{-1}$ on $U_2^- \cap U_1^+$ is

$$(y^2, \cdots, y^{n+1}) \xrightarrow{(\phi_1^+)^{-1}} \left(\sqrt{1 - \sum_{\alpha=2}^{n+1}(y^\alpha)^2}, y^2, \cdots, y^{n+1}\right)$$
$$\xrightarrow{\phi_2^-} \left(\sqrt{1 - \sum_{\alpha=2}^{n+1}(y^\alpha)^2}, y^3, \cdots, y^{n+1}\right).$$

Instead of $(y^2, \cdots, y^{n+1})$ and $(y^1, y^3, \cdots, y^{n+1})$ on U_1^+ and U_2^-, we use $(x^1, x^2, \cdots, x^n)$ and $(\overline{x}^1, \overline{x}^2, \cdots, \overline{x}^n)$, respectively. Then we have

$$\overline{x}^1 = \sqrt{1 - \sum_{i=1}^{n}(x^i)^2}, \quad \overline{x}^j = x^j, \quad j = 2, \cdots, n.$$

Hence, $\mathbb{S}^n$ is an n-dimensional smooth manifold.

Example 1.3. The *real projective space* $\mathbb{R}P^n$.

This is the set of lines of $\mathbb{R}^{n+1}$ that pass through the origin $O = (0, \cdots, 0)$. Thus, $\mathbb{R}P^n$ can be viewed as the quotient space of the unit sphere $\mathbb{S}^n = \{p \in \mathbb{R}^{n+1} \big| |p| = 1\}$ by the equivalence relation A that identifies $p \in \mathbb{S}^n$ with its antipodal point $A(p) = -p$. Indeed, each line that passes through the origin determines two antipodal points and $\mathbb{R}P^n = \mathbb{S}^n/A$.

Let $\pi : \mathbb{S}^n \to \mathbb{R}P^n$ be the canonical projection, i.e., $\pi(p) = \{p, -p\}$. By using the differentiable structure of $\mathbb{S}^n$ as in Example 1.2, one can see that $\pi(U_\alpha^+) = \pi(U_\alpha^-)$ and $\{\pi(U_\alpha^\pm)\}$ covers $\mathbb{R}P^n$. Thus, $\phi_\alpha = \phi_\alpha^\pm \circ \pi^{-1}$ is a diffeomorphism from $\pi(U_\alpha^\pm)$ to $W_\alpha \subset \mathbb{R}^n$. This gives a differentiable structure on $\mathbb{R}P^n$. Hence, $\mathbb{R}P^n$ is a smooth manifold, and $\mathbb{S}^n$ can be viewed as the two-fold covering of $\mathbb{R}P^n$.